The cubic formula

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In college algebra we make frequent use of the quadratic formula. Namely, if

$$f(x) = ax^2 + bx + c,$$

then the zeroes of $f(x)$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula can be derived by completing the square. Note that if $b^2 - 4ac$ (what we call the *discriminant*) is negative, then the quadratic polynomial $f(x)$ has two complex roots. Otherwise, you get two real roots, and in this case you don't need to know anything about complex numbers.

It is shown in upper-division math that any degree-n polynomial with rational (or real, or complex) coefficients has n complex roots. (This fact is called the *fundamental theorem of algebra.*) So, if we have a quadratic formula for finding both (possibly complex) roots of a quadratic (degree-2) polynomial, then it's natural to ask for a formula for all three roots of a cubic. Likewise, we would like a formula for all four roots of a quartic, and so on. It can be proved (the terms are *Galois theory* and *solvable groups*), that there *cannot* exist a general formula for degree 5 and above.

Here is a presentation of the cubic formula, adapted from Grove's *Algebra.* Note the following:

- It turns out that deriving this formula takes a bit more work. Details are on pages 278-279 of the reference provided below.
- The formula uses complex numbers. Even if the cubic polynomial has three real roots, some intermediate numbers in the formula are complex.
- To use the quadratic formula, you just plug in your coefficients. The cubic formula, by contrast, comes in separate steps.

- These days, it's probably easier to just do graphical root-finding using your calculator. However, it's interesting to see what people did in the old days. (The cubic formula was discovered in Renaissance Italy — for example, search Wikipedia for Nicolo Tartaglia or Scipio del Ferro).

This formula works for any cubic. At each step of the general procedure, I'll also do that step for a particular example cubic polynomial.

Step 1. Divide the cubic polynomial by its leading coefficient. For example, if you have

$$2x^3 + 18x^2 + 36x - 56,$$

then divide by the leading 2 to obtain

$$x^3 + 9x^2 + 18x - 28.$$

Now you have something of the form

$$f(x) = x^3 + ax^2 + bx + c.$$

Step 2. It turns out that it is desirable to get rid of one of the coefficients. To accomplish this, substitute

$$x = y - a/3$$

into $f(x)$ and call the result $g(y)$. For example, using $f(x)$ as above, $a = 9$ so $a/3 = 3$.

$$\begin{aligned} f(x) &= x^3 + 9x^2 + 18x - 28 \\ g(y) = f(y-3) &= (y-3)^3 + 9(y-3)^2 + 18(y-3) - 28 \\ &= y^3 - 9y^2 + 27y - 27 + 9(y^2 - 6y + 9) + 18y - 54 - 28 \\ &= y^3 - 9y^2 + 27y - 27 + 9y^2 - 54y + 81 + 18y - 54 - 28 \\ &= y^3 - 9y^2 + 9y^2 + 27y - 54y + 18y - 27 + 81 - 54 - 28 \\ &= y^3 - 9y - 28. \end{aligned}$$

Step 3. Now that we've eliminated the quadratic term, we have something of the form

$$g(y) = x^3 + py + q.$$

The next step is computing the discriminant of $g(y)$. All polynomials have discriminants, but it's particularly easy to compute now that we have only two coefficients, p and q. This is

$$D = -4p^3 - 27q^2.$$

In our example, since we have $g(y) = x^3 - 9y - 28$, we have $p = -9$ and $q = -28$. So

$$\begin{aligned} D &= -4p^3 - 27q^2 \\ &= -4(-9)^3 - 27(-28)^2 \\ &= -18252. \end{aligned}$$

Step 4. Below we'll need the numbers

$$-q/2 \qquad \text{and} \qquad \sqrt{\frac{-D}{108}},$$

so let's go ahead and compute them now. In our example, these are

$$-q/2 = 14 \qquad \text{and} \qquad \sqrt{\frac{-D}{108}} = \sqrt{\frac{18252}{108}} = \sqrt{169} = 13.$$

Step 5. Here is the formula for the three roots of the cubic $g(y)$:

$$\begin{aligned} y_1 &= u_1 + v_1 \\ y_2 &= \omega u_1 + \omega^2 v_1 \\ y_3 &= \omega^2 u_1 + \omega v_1. \end{aligned}$$

We need to know what u_1, v_1, ω, and ω^2 are. First, ω and ω^2 are constants:

$$\begin{aligned} \omega &= \frac{-1 + i\sqrt{3}}{2} \\ \omega^2 &= \frac{-1 - i\sqrt{3}}{2} \end{aligned}$$

(Side note: ω, ω^2, and 1 are the three complex numbers whose cube is 1. Try FOILing out the product $\omega \cdot \omega^2$.) Also,

$$\begin{aligned} u_1 &= \sqrt[3]{-q/2 + \sqrt{-D/108}}, \\ v_1 &= \sqrt[3]{-q/2 - \sqrt{-D/108}}. \end{aligned}$$

In our example, we have

$$\begin{aligned} u_1 &= \sqrt[3]{14 + 13} = \sqrt[3]{27} = 3; \\ v_1 &= \sqrt[3]{14 - 13} = \sqrt[3]{1} = 1. \end{aligned}$$

So the first root is

$$y_1 = 3 + 1 = 4.$$

The second root is

$$
\begin{aligned}
y_2 &= 3\left(\frac{-1+i\sqrt{3}}{2}\right)+\left(\frac{-1-i\sqrt{3}}{2}\right) \\
&= \left(\frac{-3+3i\sqrt{3}}{2}\right)+\left(\frac{-1-i\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(-3+3i\sqrt{3}-1-i\sqrt{3}\right) \\
&= \frac{1}{2}\left(-3-1+3i\sqrt{3}-i\sqrt{3}\right) \\
&= \frac{1}{2}\left(-4+2i\sqrt{3}\right) \\
&= -2+i\sqrt{3}.
\end{aligned}
$$

The third root is

$$
\begin{aligned}
y_3 &= 3\left(\frac{-1-i\sqrt{3}}{2}\right)+\left(\frac{-1+i\sqrt{3}}{2}\right) \\
&= \left(\frac{-3-3i\sqrt{3}}{2}\right)+\left(\frac{-1+i\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\left(-3-3i\sqrt{3}-1+i\sqrt{3}\right) \\
&= \frac{1}{2}\left(-3-1-3i\sqrt{3}+i\sqrt{3}\right) \\
&= \frac{1}{2}\left(-4-2i\sqrt{3}\right) \\
&= -2-i\sqrt{3}.
\end{aligned}
$$

Step 6. We just found the roots of $g(y)$. To finish up, we need to undo the change of variable

$$x = y - a/3.$$

In our example, we found

$$y = 4, \quad -2+i\sqrt{3}, \quad \text{and} \quad -2-i\sqrt{3}.$$

Since $a/3$ was 3, we have, for the original polynomial

$$2x^3 + 18x^2 + 36x - 56,$$

the three roots

$$x = 1, \quad -5+i\sqrt{3}, \quad \text{and} \quad -5-i\sqrt{3}.$$

Reference

L.C. Grove, *Algebra*, Dover, 2004.