Derivation of sum and difference identities for sine and cosine

John Kerl

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The authors of your trigonometry textbook give a geometric derivation of the sum and difference identities for sine and cosine. I find this argument unwieldy — I don't expect you to remember it; in fact, I don't remember it. There's a standard algebraic derivation which is far simpler. The only catch is that you need to use complex arithmetic, which we don't cover in Math 111. Nonetheless, I will present the derivation so that you will have seen how simple the truth can be, and so that you may come to understand it after you've had a few more math courses. And in fact, all you need are the following facts:

- *Complex numbers* are of the form $a+bi$, where a and b are real numbers and i is defined to be a square root of -1. That is, $i^2 = -1$. (Of course, $(-i)^2 = -1$ as well, so $-i$ is the other square root of -1.)
- The number a is called the *real part* of $a + bi$; the number b is called the *imaginary part* of $a + bi$. All the real numbers you're used to working with are already complex numbers — they simply have zero imaginary part.
- To add or subtract complex numbers, add the corresponding real and imaginary parts. For example, $2 + 3i$ plus $4 + 5i$ is $6 + 8i$.
- To multiply two complex numbers $a + bi$ and $c + di$, just FOIL out the product $(a + bi)(c + di)$ and use the fact that $i^2 = -1$. Then collect like terms.
- The familiar exponential function $f(x) = e^x$ takes real-valued input. However, it can be extended to take complex-valued input. All the usual rules for exponents apply, so

  $$e^{a+bi} = e^a e^{bi}.$$

  We compute e^a as always — this is the same exponential function as always. The question is, what does it mean to raise e to an imaginary power? I assert to you that we write

  $$e^{bi} = \cos(b) + i \sin(b)$$

where the cosine and sine functions are as usual. This famous formula is called *Euler's formula* (*Euler* is pronounced *Oiler*). You can read all about this formula on Wikipedia — also see their nice article on the complex numbers.

Given these facts, we can simply write down what $e^{i(\alpha+\beta)}$ is: the sum and difference formulas for sine and cosine fall out as a consequence. Using the usual rules for exponents, we can write this as

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}.$$

Now all we need to do is write out the two sides using Euler's formula. The left-hand side is

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

Using the definition, FOILing, and collecting like terms, the right-hand side is

$$\begin{aligned} e^{i\alpha}e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta). \end{aligned}$$

Equating real and imaginary parts of the left-hand side and the right-hand side gives us, two for the price of one, the familiar sum identities for sine and cosine:

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}$$

Repeat this for $e^{i(\alpha-\beta)}$ to get the difference identities. You can do that — just remember that cosine and sine are even and odd functions, respectively, so $\cos(-\beta) = \cos(\beta)$ and $\sin(-\beta) = -\sin(\beta)$.

In summary, we have:

$$\begin{aligned} \sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \end{aligned}$$