The gamma-integral formula

John Kerl

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This is an often-recurring formula which I thought I'd finally jot down for handy reference. It's a special case of the gamma function which comes up in probability and statistics, as well as introductory calculus.

The claim is that

$$\int_0^\infty x^n e^{-x}\, dx = n!.$$

Proof. First take $n = 0$. Then

$$\int_0^\infty e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1.$$

Now define

$$I_n = \int_0^\infty x^n e^{-x}\, dx.$$

For $n \geq 1$, integrate by parts. Put

$$\begin{aligned} u &= x^n & dv &= e^{-x}\, dx \\ du &= nx^{n-1} & v &= -e^{-x}. \end{aligned}$$

Then

$$\begin{aligned} I_n &= \int_0^\infty u\, dv \\ &= [uv]_0^\infty - \int_0^\infty v\, du \\ &= \left[-x^n e^{-x}\right]_0^\infty + n \int_0^\infty x^{n-1} e^{-x}\, dx \\ &= 0 + nI_{n-1}. \end{aligned}$$

This recurrence relation defines $n!$ so we have

$$\int_0^\infty x^n e^{-x}\, dx = n!.$$

□

The **gamma function** is defined to be the value of this integral, even when n is not necessarily an integer:

$$\Gamma(z) := \int_0^\infty x^{z-1} e^{-x}\, dx.$$

(This blows up when z is zero or a negative integer.) So, we have

$$\Gamma(n) = (n-1)!.$$