The exemplar model in mathematics instruction

John Kerl

November 2, 2007

I was born not knowing and have had only a little time to change that here and there.
— Richard Feynman (1918-1988).

I am no math historian; in fact, as a second-year graduate student and a teacher with only two years' classroom experience, I am only a budding mathematician. Yet this much seems to me to be a clear pattern in the development of mathematics:

- Real-world problems are studied; mathematical models are developed.
- After some time goes by, wherein mathematicians collaborate and communicate, similarities are discovered between problems which had seemed distinct. (For example, permutations on symbols of letters and Galois' original "substitutions"; systems of linear equations and systems of differential equations.)
- A system of axioms is developed, encapsulating and abstracting the common features of the once-disparate systems. (In the above examples, we obtained abstract groups and vector spaces, respectively.)
- We then study the abstract systems, proving general results and developing general methods which apply to all the original specific situations, *and* any others which might apply in the future. Herein lies much of the power of mathematics: once we are familiar with, say, abstract groups, whenever we encounter a brand-new object, if we recognize it is a group then we instantly know much about it.
- After enough time has gone by, we begin to teach the abstractions first. In a pure-math environment, one might forget (or forget to mention) the origins entirely.

So much for the *phylogeny* of mathematics — its evolutionary development across the lives of many thinkers. What about its *ontogeny* — the development of ideas within the mind of a single thinker? Like Feynman, none of us were born knowing calculus; every mathematics student and future mathematician must painstakingly relearn these ideas from scratch.

My claim is that **ontogeny recapitulates phylogeny**. This is a biological maxim[1], reflecting the fact that the embryonic development of an individual retraces (albeit inexactly) the evolutionary development of that individual's species. When I was an embryo, very early on I had gill slits — and so did you. One pair of slits became our ears; the rest closed. One might think that we, as modern humans, could reproduce more humans directly, pole-vaulting over the intermediate stages — but we do not.

Likewise in the development of ideas. When we are young, we experience many specific situations. We later learn to abstract and categorize: many once-different objects begin to go by the common name of *fork* or

[1] Attributed to the zoologist Ernst Häckel. The subject area is called *recapitulation theory*.

dog. (See the Boas quote below.) One might think we could pole-vault over our early development, but we do not.

Few would argue that we could hand a two-year-old a dictionary and have him or her skip over those early learning experiences. But in the mathematics classroom — blinded by the power and beauty of modern methods — we often do just that. In my eyes, the key trait distinguishing the mathematics of the 20th and 21st centuries from that of the centuries before is the *axiomatic approach.* This is a rightful component of our discipline. Yet we start the first day of an upper-division or graduate algebra course with a mantra of the form "A *group* is a set endowed with a binary operation ..."; we begin a geometry course with an abstract discussion of atlases and manifolds. This feels good and it feels right, especially to the teacher: we are going straight for the beauty and the elegance. It is especially easy for the teacher, who has already seen these concepts put to use in dozens of concrete situtations: the teacher is ready for abstraction.

But when we do this, we leave our students behind. We attempt to leap over the *absolutely necessary* instantiation of ideas that we ourselves went through (whether we did so consciously or not). If we are to retain students in mathematics, and if modern pure mathematics is to remain relevant to the rest of the world, we must explicitly acknowledge that ontogeny recapitulates phylogeny. This does not mean that we need to retrace the full development of every mathematical concept, with all its false starts and wrong turns. It does mean, however, that we *first* need to tell our students *why* a discipline was invented in the first place. We need to present the applied problems which gave birth to the subject we are teaching. We must give our students the opportunity to generalize from concrete situations.

One hears the objection: *I don't have time for such trivialities. Students should be able to do this on their own.* I reply, as a graduate student and teacher: It is hard enough for our students to learn what we teach them; it is far harder for them to learn what we do not teach them. The training of a new mind is something worth doing right: it is better that it happen, not in spite of our teaching methods, but *because* of them.

Suppose that you want to teach the "cat" concept to a very young child. Do you explain that a cat is a relatively small, primarily carnivorous mammal with retractible claws, a distinctive sonic output, etc.? I'll bet not. You probably show the kid a lot of different cats, saying "kitty" each time, until it gets the idea. To put it more generally, generalizations are best made by abstraction from experience. — R. P. Boas