What's with the base?

John Kerl

January 12, 2007

Suppose I ask you to come up with an "exponential model" for the following data: You have a bank account with \$1000 initial balance and \$4000 after 20 years. Well, there are several choices for an "exponential model":

$$P = P_0 e^{rt}, \qquad P = P_0 2^{kt}, \qquad P = P_0 a^t.$$

(I'm letting t be the time in years, P be the account balance, P_0 be the initial deposit, and r, k, and a being constants to be found.)

It seems like these three different models give us different answers. So, what gives? Let's try them and find out.

1 The e^{rt} model

Let's write down what we know: $P(0) = 1000$ and $P(20) = 4000$. This gives us two equations:

$$\begin{aligned} 1000 &= P_0 e^{0 \cdot r} = P_0 \\ 4000 &= P_0 e^{20 \cdot r}. \end{aligned}$$

From the first equation we can read off $P_0 = 1000$, and then we can plug that into the second equation to get

$$\begin{aligned} 4000 &= 1000 \cdot e^{20r} \\ 4 &= e^{20r} \end{aligned}$$

Since r is up in an exponent, we can take logs of both sides to bring it down:

$$\begin{aligned} \ln(4) &= \ln(e^{20r}) \\ 20r &= \ln(4) \\ r &= \ln(4)/20 \approx 0.0693. \end{aligned}$$

This means our first model for the bank-account data is

$$P = 1000 \cdot e^{0.0693t}.$$

2 The 2^{kt} model

Again writing down what we know gives us two equations:

$$\begin{aligned} 1000 &= P_0 2^{0\cdot k} = P_0 \\ 4000 &= P_0 2^{20\cdot k}. \end{aligned}$$

From the first equation we can again read off $P_0 = 1000$, and then we can plug that into the second equation to get

$$\begin{aligned} 4000 &= 1000 \cdot 2^{20k} \\ 4 &= 2^{20k} \end{aligned}$$

We can take logs just as before; that would work fine. Here, though, since $4 = 2^2$, we can instead do

$$2^2 = 2^{20k}$$

from which (since the exponential functions are 1-1) we can equate the exponents:

$$\begin{aligned} 2 &= 20k \\ k &= 0.1. \end{aligned}$$

So our second model for the bank-account data is

$$P = 1000 \cdot 2^{0.1t}.$$

3 The a^t model

Yet again the specified data give us two equations:

$$\begin{aligned} 1000 &= P_0 a^0 = P_0 \\ 4000 &= P_0 a^{20}. \end{aligned}$$

We have $P_0 = 1000$ as usual from the first equation. Plugging that into the second equation we have

$$\begin{aligned} 4000 &= 1000a^{20} \\ 4 &= a^{20} \\ a &= \sqrt[20]{4} = 4^{1/20} \approx 1.0718. \end{aligned}$$

So our third model for the bank-account data is

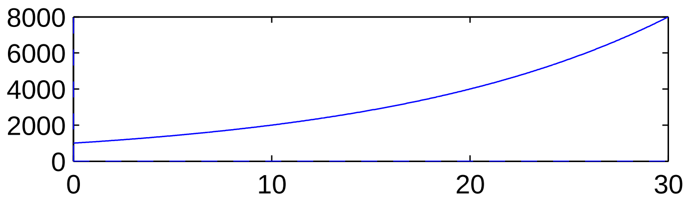
$$P = 1000 \cdot 1.0718^t.$$

4 Do they all work?

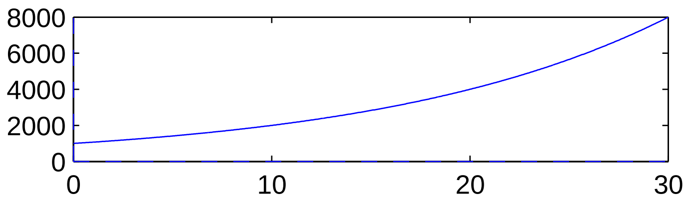
To summarize, here are three models for the account balance with \$1000 initial balance and \$4000 after 20 years:

$$\begin{aligned} P &= 1000 \cdot e^{0.0693t} \\ P &= 1000 \cdot 2^{0.1t} \\ P &= 1000 \cdot 1.0718^t. \end{aligned}$$

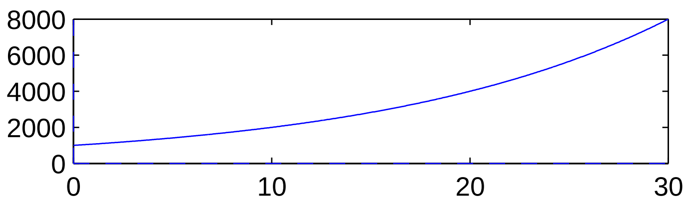
Here is a graph of the first model:



Here is a graph of the second model:



Here is a graph of the third model:



It looks like they're all doing the same job. But how can different bases do the same thing?

5 Why do they all work?

The key insight is the following property of exponents:

$$\begin{aligned} e^{rt} &= (e^r)^t \\ 2^{kt} &= (2^k)^t. \end{aligned}$$

This means in particular that we can rewrite our three models as:

$$\begin{aligned} P &= 1000(e^{0.0693})^t \\ P &= 1000(2^{0.1})^t \\ P &= 1000(1.0718)^t. \end{aligned}$$

So, what are those three numbers in parentheses? They are

$$e^{0.0693} = 2^{0.1} = 1.0718.$$

What we're seeing is evidence that even if we change the base of the exponential model, the constant k or r changes to compensate.

We can choose a particular base if we know (or guess) ahead of time that it will work out neatly. But even if we choose a different base, it will still work.

Some points to be made:

- The base-2 model seems to be a nice choice here — I set up this problem so that there was \$1000 at 0 years and \$4000 at 20 years, so \$2000 at 10 years. That is, the doubling time for the investment is 10 years.

  (It turns out that even if you don't use base 2, all exponential models have constant doubling time. A nice exercise would be to convince yourself of this.)

- The base-a model is nice because it tells you by what percent your money grows each year: $a = 1.0718$, so after one year, you have 0.0718 more than initially — that is, 7.18% more.

- The base-e model is familiar from the compound interest formula in other courses.

- Notice that the r in the base-e model is about 7% — more precisely, we found $r = 6.931\%$. Also the a is about 107% — we found 107.18%. That is, it looks like $e^r \approx 1 + r$. This is no coincidence — you'll see why when you get to Taylor series in second-semester calculus.