Poisson summation and the discrete Fourier transform

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Abstract

Riemann sums and counting measure are used to motivate an intuitive derivation of the discrete Fourier transform. Poisson summation is then used to obtain an error estimate for the DFT with respect to the continuous Fourier transform on the circle. This paper fulfills a term-paper requirement for Bill Faris' spring 2006 Real Analysis course (Math 523B).

Contents

1 Fourier transforms and Fourier series

Faris [**Far**] points out that the general context of Fourier analysis is an abelian group and its dual group. (The duality is in the Pontryagin sense, the details of which are beyond the scope of this paper.) The elements of the group are typically thought of as time or space variables; the elements of the dual group are thought of as angular-frequency or wave-number variables. The following may be shown to be dual pairs of abelian groups:

(I) The real line and itself. In this case, we have Fourier transforms: given $f \in L^2(\mathbb{R}, dx)$, we compute $\hat{f} \in L^2(\mathbb{R}, dk/2\pi)$.

(II) The unit circle and the integers $\mathbb{Z}$. In this case, we have Fourier series: given $f \in L^2(T, dx)$, we compute $\mathbf{c} \in \ell^2(\mathbb{Z})$.

(III) The integers $\mathbb{Z}$ and the unit circle. This is the case for using Fourier series to reconstruct the original function.

(IV) A finite cyclic group of order N and itself. Here, we have discrete Fourier transforms.

This paper elaborates on transforms of type IV. One may define the discrete Fourier transform by fiat. Instead, though, we consider two motivating constructions.

2 The discrete Fourier transform via Riemann sums

Consider an approximation to a transform of type II. Define T_L to be the circle of circumference L. Let $f(x)$ be continuous on T_L. Equivalently, think of $f(x)$ as being periodic on the real line, with period L.

The basis functions are of the form e^{ikx}, for some values of k. To see what these values must be, look at $x = L$. For e^{ikx} to have period L, kL must be an integer multiple of 2π. That is, we take

$$k = \frac{2\pi\ell}{L}$$

for $\ell \in \mathbb{Z}$.

The Fourier coefficients $\hat{f}(k)$ are

$$\hat{f}(k) = \langle e^{ikx}, f(x)\rangle = \frac{1}{L}\int_0^L e^{-ikx} f(x)dx.$$

Form a Riemann sum with N evenly spaced mesh points, deferring for a moment the well-definedness of the sum on $L^2(T_L, dx)$. We have

$$x_j = j\Delta x$$

where

$$\Delta x = \frac{L}{N}.$$

Then the integral is approximated by

$$\frac{1}{L}\int_0^L e^{-ikx} f(x)dx \approx \frac{1}{L}\sum_{j=0}^{N-1} e^{-ikx_j} f(x_j)\Delta x.$$

Now recall that the angular frequencies k were also evenly spaced, namely, with

$$k = \ell \Delta k \quad \text{and} \quad \Delta k = \frac{2\pi}{L}.$$

Thus the Δx and Δk are not unrelated. In fact,

$$\Delta k \Delta x = \frac{L}{N} \cdot \frac{2\pi}{L} = \frac{2\pi}{N}.$$

Making substitutions, we now have for the estimated integral

$$\begin{aligned} \hat{f}(k) &\approx \frac{1}{L}\sum_{j=0}^{N-1} e^{-ikx_j} f(x_j)\Delta x \\ &= \frac{1}{L}\sum_{j=0}^{N-1} e^{-i\ell\Delta k j\Delta x} f(x_j)\Delta x \\ &= \frac{1}{L}\sum_{j=0}^{N-1} e^{-i\ell\frac{2\pi}{L}j\frac{L}{N}} f(x_j)\frac{L}{N} \\ &= \frac{1}{N}\sum_{j=0}^{N-1} e^{-i2\pi\ell j/N} f(x_j). \end{aligned} \tag{2.1}$$

Observe that now there are N input points, $f(x_0)$ through $f(x_{N-1})$. Also observe that, when restricted to the mesh $x = j\Delta x$, the basis functions

$$e^{ikx} = e^{i2\pi\ell j/N}$$

have identical values when ℓ is shifted by multiples of N. Thus the periodicity of the coefficients arises naturally. There are N input points and N output points. Phrased differently, a pure sinusoidal input of frequency 1 is indistinguishable from a pure sinusoidal input of frequency $N+1$ when the input data are sampled only at the N mesh points. (This fact has significant repercussions in the implementation of signal-processing systems. See [**Lyons**] for more information.)

An advantage of this derivation of the discrete Fourier transform is that it uses elementary and easily visualizable Riemann sums. One asks, however, about well-definedness. Since $f \in C(T_L)$, and since T_L is compact, $f \in L^2(T_L, dx)$. Likewise, $f \in L^1(T_L, dx)$. Due to the continuity, we may without loss of generality take f to be a particular representative for an equivalence class with respect to Lebesgue measure on T_L.

An alternative derivation of the discrete Fourier transform may be obtained by defining a counting measure χ on T_L via

$$\chi(f) = \frac{1}{N}\sum_{j=0}^{N-1} f(x_j)$$

using the same mesh $x = j\Delta x$ as before. This approach results in the same DFT formula, namely, equation 2.1.

3 Poisson summation on the real line

In section 6, Poisson summation will be used to analyze approximation error in the DFT. First, however, the theory will be examined on the real line, to fix concepts. The presentation here elaborates on section 24.6 of [**Far**].

Proposition 3.1. *Let L be a positive real number. Let $f(x) \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$ with Fourier transform $\hat{f} \in L^1(\mathbb{R}, dk/2\pi)$. Further suppose that*

$$\sum_{k \in \mathbb{Z}\frac{2\pi}{L}} \hat{f}(k) < +\infty.$$

Then

$$\sum_{x \in \mathbb{Z}L} f(x) = \frac{1}{L} \sum_{k \in \mathbb{Z}\frac{2\pi}{L}} \hat{f}(k).$$

Proof. Recall that for $k \in \mathbb{R}$ we have

$$\hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx.$$

Now define (this utility function being the central trick of the proof)

$$S(t) = \sum_{x \in \mathbb{Z}L} f(x+t).$$

This sum is finite since f is in $L^1(\mathbb{R}, dx)$. Since $S(t)$ is well-defined on $\mathbb{R}/\mathbb{Z}L = T_L$, it has Fourier coefficients

$$a_k = \frac{1}{L} \int_0^L e^{-ikt} S(t) dt$$

for $k \in \mathbb{Z}\frac{2\pi}{L}$. We may reconstruct $S(t)$ via

$$S(t) = \sum_{k \in \mathbb{Z}\frac{2\pi}{L}} a_k e^{ikt}. \tag{3.1}$$

But

$$a_k = \frac{1}{L} \int_0^L e^{-ikt} S(t) dt = \frac{1}{L} \int_0^L e^{-ikt} \left[\sum_{x \in \mathbb{Z}L} f(x+t) \right] dt.$$

Since $f \in L^1(\mathbb{R}, dx)$, by Fubini's theorem we may interchange the integral and sum to obtain

$$a_k = \frac{1}{L} \sum_{x \in \mathbb{Z}L} \int_0^L e^{-ikt} f(x+t) dt.$$

Since $x \in \mathbb{Z}L$, let $x = jL$ for $j \in \mathbb{Z}$. Using change of variable $u = jL + t$ in the integrand, and recalling that we are restricting k to be in $\mathbb{Z}\frac{2\pi}{L}$, we have

$$\begin{aligned}
a_k &= \frac{1}{L} \sum_{j=-\infty}^{+\infty} \int_{t=0}^{t=L} e^{-ikt} f(jL+t) dt \\
&= \frac{1}{L} \sum_{j=-\infty}^{+\infty} \int_{u=jL}^{u=(j+1)L} e^{-ik(u-jL)} f(u) du \\
&= \frac{1}{L} \sum_{j=-\infty}^{+\infty} \int_{u=jL}^{u=(j+1)L} e^{-iku} f(u) du \\
&= \frac{1}{L} \int_{u=-\infty}^{u=+\infty} e^{-iku} f(u) du \\
&= \frac{\hat{f}(k)}{L}.
\end{aligned}$$

Now simply evaluate $S(t)$ at $t = 0$. We have

$$S(t) = \sum_{x \in \mathbb{Z}L} f(x+t)$$

and

$$S(t) = \sum_{k \in \mathbb{Z}\frac{2\pi}{L}} a_k e^{ikt} = \frac{1}{L} \sum_{k \in \mathbb{Z}\frac{2\pi}{L}} \hat{f}(k) e^{ikt}$$

so

$$S(0) = \sum_{x \in \mathbb{Z}L} f(x) = \frac{1}{L} \sum_{k \in \mathbb{Z}\frac{2\pi}{L}} \hat{f}(k).$$

□

4 Poisson summation on the circle

The Poisson summation formula was worked out on the (familiar) real line in order to give a concrete example for the more abstract setting. Recall from section 1 that in this paper we are taking Pontryagin duals as given. In particular, if G is an abelian group, write $\hat{G}$ for its dual. In the previous section, we had $G = \mathbb{R}$ with closed subgroup $H = \mathbb{Z}L$ and quotient $G/H = T_L$. Now, the dual group for G is $\hat{G} = \mathbb{R}$, and the duals of H and G/H are as before in this paper, namely:

Group			**Dual**		
G	$=$	$\mathbb{R}$	$\hat{G}$	$=$	$\mathbb{R}$
H	$=$	$\mathbb{Z}L$	$\hat{H}$	$=$	$T_{2\pi/L}$
G/H	$=$	T_L	$\widehat{G/H}$	$=$	$\mathbb{Z}\frac{2\pi}{L}$

Given this terminology, we can walk back through the proof of the Poisson summation formula on the real line, which will suggest how to correctly write the summation formula for the circle. We took $f : G \to \mathbb{R}$, with $\hat{f} : \hat{G} \to \mathbb{R}$. We required

$$\sum_{k \in \widehat{G/H}} \hat{f}(k) < +\infty,$$

and we claimed that

$$\sum_{x \in H} f(x) = \sum_{k \in \widehat{G/H}} \hat{f}(k).$$

The key to the proof was that e^{-ikx} was well-defined on the quotient space $\widehat{G/H}$, and that G was the union of translates of G/H. Now we simply replace G, H, et al. by the following, using the information about Δx and Δk obtained in section 2:

Group			**Dual**		
G	$=$	$\mathbb{R}/\mathbb{Z}L$	$\hat{G}$	$=$	$\mathbb{Z}\frac{2\pi}{L}$
	$=$	T_L			
H	$=$	$\mathbb{Z}\frac{L}{N}/\mathbb{Z}L$	$\hat{H}$	$=$	$\mathbb{Z}_N\frac{2\pi}{L}$
	$=$	$\mathbb{Z}_N\left(\frac{L}{N}\right)$			
G/H	$=$	$\frac{\mathbb{R}/\mathbb{Z}L}{\mathbb{Z}\frac{L}{N}/\mathbb{Z}L}$	$\widehat{G/H}$	$=$	$\mathbb{Z}\frac{2\pi N}{L}$
	$=$	$\mathbb{R}/\mathbb{Z}\frac{L}{N}$			
	$=$	$T_{L/N}$			

Proposition 4.1. *Let L be a positive real number. Let $f(x) \in L^1(T_L, dx)$ with Fourier transform $\hat{f} \in L^1(\mathbb{Z}\frac{2\pi}{L}, dk/2\pi)$. Further suppose that*

$$\sum_{k \in \mathbb{Z}\frac{2\pi N}{L}} \hat{f}(k) < +\infty.$$

Then

$$\sum_{x \in \mathbb{Z}_N \frac{L}{N}} f(x) = N \sum_{k \in \mathbb{Z}\frac{2\pi N}{L}} \hat{f}(k).$$

Proof. For $k \in \mathbb{Z}\frac{2\pi}{L}$ we have

$$\hat{f}(k) = \int_0^L e^{-ikx} f(x) dx.$$

Define

$$S(t) = \sum_{x \in \mathbb{Z}_N \frac{L}{N}} f(x+t).$$

Since $S(t)$ is well-defined on the quotient $T_{L/N}$, it has Fourier coefficients

$$a_k = \frac{N}{L} \int_0^{L/N} e^{-ikt} S(t) dt$$

for $k \in \mathbb{Z}\frac{2\pi N}{L}$. We may reconstruct $S(t)$ via

$$S(t) = \sum_{k \in \mathbb{Z}\frac{2\pi N}{L}} a_k e^{ikt}. \tag{4.1}$$

But

$$a_k = \frac{N}{L} \int_0^{L/N} e^{-ikt} S(t) dt = \frac{N}{L} \int_0^{L/N} e^{-ikt} \left[\sum_{x \in \mathbb{Z}_N \frac{L}{N}} f(x+t) \right] dt.$$

Since $f \in L^1(T_L, dx)$, by Fubini's theorem we may again interchange the integral and sum to obtain

$$a_k = \frac{N}{L} \sum_{x \in \mathbb{Z}_N \frac{L}{N}} \int_0^{L/N} e^{-ikt} f(x+t) dt.$$

Since $x \in \mathbb{Z}_N \frac{L}{N}$, let $x = jL/N$ for $j \in \mathbb{Z}_N$. Using change of variable $u = jL/N + t$ in the integrand, and recalling that we are restricting k to be in $\mathbb{Z}\frac{2\pi N}{L}$, we have

$$\begin{aligned}
a_k &= \frac{N}{L} \sum_{j=0}^{N-1} \int_{t=0}^{t=L/N} e^{-ikt} f(jL/N + t) dt \\
&= \frac{N}{L} \sum_{j=0}^{N-1} \int_{u=jL/N}^{u=(j+1)L/N} e^{-ik(u-jL/N} f(u) du \\
&= \frac{N}{L} \sum_{j=0}^{N-1} \int_{u=jL/N}^{u=(j+1)L/N} e^{-iku} f(u) du \\
&= \frac{N}{L} \int_{u=0}^{u=L} e^{-iku} f(u) du \\
&= N \hat{f}(k).
\end{aligned}$$

Evaluating $S(t)$ at $t = 0$, we have

$$S(t) = \sum_{x \in \mathbb{Z}_N \frac{L}{N}} f(x+t)$$

and

$$S(t) = \sum_{k \in \mathbb{Z} \frac{2\pi N}{L}} a_k e^{ikt} = N \sum_{k \in \mathbb{Z} \frac{2\pi N}{L}} \hat{f}(k) e^{ikt}$$

so

$$S(0) = \sum_{x \in \mathbb{Z}_N \frac{L}{N}} f(x) = N \sum_{k \in \mathbb{Z} \frac{2\pi N}{L}} \hat{f}(k).$$

□

5 The Poisson summation formula for the DFT

Now apply proposition 4.1 not to $f(x)$ but rather to $e^{-ipx} f(x)/N$.

A little lemma shows that a phase shift in the time domain is a translation in the frequency domain. To see this, compute:

$$\frac{1}{L} \int_0^L e^{-ikx} \left[e^{-ipx} f(x) \right] dx = \frac{1}{L} \int_0^L e^{-i(k+p)x} f(x) dx = \hat{f}(k+p).$$

By the lemma and proposition 4.1, we have

$$\frac{1}{N} \sum_{x \in \mathbb{Z}_N \frac{L}{N}} e^{-ipx} f(x) = \sum_{k \in \mathbb{Z} \frac{2\pi N}{L}} \hat{f}(k+p). \tag{5.1}$$

The left-hand side is precisely the N-point discrete Fourier transform, at p, of $f(x)$ on T_L. For brevity, call this $\hat{f}_D(p)$. Then equation 5.1 becomes

$$\boxed{\hat{f}_D(p) = \sum_{k \in \mathbb{Z} \frac{2\pi N}{L}} \hat{f}(k+p).} \tag{5.2}$$

We may summarize the Poisson formula in the following way: The N-point DFT folds energy from all frequencies which are congruent mod N into the same bin. Thus the error in the DFT is precisely the amount of energy at frequencies outside the range $k = -N/2, \ldots, N/2-1$. The Riemann-Lebesgue lemma [**Far**] assures us that for $f \in L_1$, the Fourier coefficients $\hat{f}(p)$ eventually go to zero. However, as yet we lack a bound on how fast they go to zero. In particular, consider a sinusoidal input with frequency $aN+1$, for any integer a. Then the N-point DFT will fold all the input's energy into $\hat{f}_D(1)$. Yet, the true Fourier coefficient will have its energy at $\hat{f}(aN+1)$. This high-frequency signal is highly oscillatory, though, which is to say that its second derivative takes on large values. This example motivates the following error analysis.

6 First-order error analysis

Compare the right-hand side of equation 5.2 to the continuous Fourier transform

$$\hat{f}(p) = \frac{1}{L} \int_0^L e^{-ipx} f(x) dx$$

for $p \in \mathbb{Z}\frac{2\pi}{L}$. The error in the DFT with respect to the CFT is

$$\left|\hat{f}(p) - \hat{f}_D(p)\right| = \left|\hat{f}(p) - \sum_{k \in \mathbb{Z}\frac{2\pi N}{L}} \hat{f}(k+p)\right| = \left|\sum_{k \in \mathbb{Z}\frac{2\pi N}{L}, k \neq 0} \hat{f}(k+p)\right|.$$

Proposition 6.1. *The error in the DFT depends linearly on the upper bound on the second derivative of the input. Furthermore, the convergence of the DFT to the true Fourier transform is quadratic in the number of mesh points.*

Proof. Suppose $f(x)$ has bounded second derivative, i.e. let M_2 be a positive real number such that for all $x \in T_L$,

$$|f''(x)| \leq M_2. \tag{6.1}$$

To relate this bound on the second derivative to a bound on $\hat{f}(k+p)$ for $k \neq 0$, use the derivative property of the Fourier transform. Letting $q = k + p$ for brevity, this is

$$\widehat{f'}(q) = iq\hat{f}(q). \tag{6.2}$$

A straightforward computation involving the transform $\widehat{f'}(q)$ yields a harmonic series, which is divergent. Note, however, that the square harmonic series converges. Thus, a slight trick is called for. Since we are assuming $f \in C(T_L)$, we have $f \in L^2(T_L, dx)$. Following the technique of theorem 23.11 of [**Far**], put $1 = (k+p)/(k+p)$ and use the Cauchy-Schwarz inequality on ℓ^2 to obtain

$$\begin{aligned} \left|\sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} \hat{f}(k+p)\right| &= \left|\sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} \frac{1}{k+p}(k+p)\hat{f}(k+p)\right| \\ &\leq \sqrt{\sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} \frac{1}{(k+p)^2}} \sqrt{\sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} |(k+p)\hat{f}(k+p)|^2} \end{aligned}$$

The use of the inequality is justified since we take $f \in C(T_L)$, hence $f \in L^2(T_L, dx)$, and therefore $\hat{f} \in L^2(T_L, dk/2\pi)$.

For the first term, since $k \in \mathbb{Z}\frac{2\pi N}{L}$, take $k = \ell 2\pi N/L$. Then

$$\sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} \frac{1}{k^2} = \sum_{\ell \neq 0, \ell \in \mathbb{Z}} \left(\frac{L}{\ell 2\pi N}\right)^2 = \frac{2L^2}{4\pi^2 N^2} \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} = \frac{L^2}{2\pi^2 N^2} \frac{\pi^2}{6} = \frac{L^2}{12N^2}.$$

For the other term in the error estimate, by equation 6.2,

$$\sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} |(k+p)\hat{f}(k+p)|^2 = \sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} |\widehat{f'}(k+p)|^2 = \sum_{k \neq 0, k \in \mathbb{Z}\frac{2\pi N}{L}} \left|\frac{\widehat{f''}(k+p)}{k+p}\right|^2$$

To relate $\widehat{f''}(k+p)$ to the known bound on $f''(x)$, we may use the following lemma. Let $g(x) \in C(T_L)$. By the compactness of T_L, $g(x)$ is bounded: say by a positive real constant M. Then $|\hat{g}(q)| \leq M$. To see this, compute

$$\begin{aligned} |\hat{g}(q)| &= \left|\frac{1}{L}\int_0^L e^{-iqx} g(x)dx\right| \leq \frac{1}{L}\int_0^L |e^{-iqx} g(x)|dx = \frac{1}{L}\int_0^L |e^{-iqx}||g(x)|dx \\ &= \frac{1}{L}\int_0^L |g(x)|dx \leq \frac{1}{L}\int_0^L M dx = M. \end{aligned}$$

Apply the lemma to the second term in the error estimate:

$$\begin{aligned}
\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \left|\frac{\widehat{f''}(k+p)}{k+p}\right|^2 &\leq \sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \left|\frac{M_2}{k+p}\right|^2 = M_2^2 \sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \left|\frac{1}{k+p}\right|^2 \\
&\leq M_2^2 \sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \frac{1}{k^2} = \frac{M_2^2 L^2}{12N^2}.
\end{aligned}$$

Combining these results yields

$$\begin{aligned}
\left|\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \hat{f}(k+p)\right| &\leq \sqrt{\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \frac{1}{(k+p)^2}} \sqrt{\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} |(k+p)\hat{f}(k+p)|^2} \\
&\leq \frac{L}{N\sqrt{12}} \sqrt{\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} |(k+p)\hat{f}(k+p)|^2} \\
&= \frac{L}{N\sqrt{12}} \sqrt{\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} |\widehat{f'}(k+p)|^2} \\
&= \frac{L}{N\sqrt{12}} \sqrt{\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \left|\frac{\widehat{f''}(k+p)}{k+p}\right|^2} \\
&\leq \frac{LM_2}{N\sqrt{12}} \sqrt{\sum_{k\neq 0, k\in \mathbb{Z}\frac{2\pi N}{L}} \frac{1}{k^2}} \\
&= \frac{L^2 M_2}{12N^2}.
\end{aligned}$$

$\square$

7 Higher-order error analysis

Now suppose that f is $m+1$ times differentiable. Then its $(m+1)$st derivative is continuous on the compact set T_L, and hence is in $L^2(T_L, dx)$. Furthermore, $f^{(m+1)}(x)$ is bounded, say by M_{m+1}.

Proposition 7.1. *The error in the DFT depends linearly on the upper bound on the $(m+1)$st derivative of the input. Furthermore, the convergence of the DFT to the true Fourier transform is inversely proportional to N^{m+1} where N is the number of mesh points.*

Proof. We may repeatedly apply equation 6.2 to obtain

$$
\begin{aligned}
\left|\sum_{k\neq 0, k\in\mathbb{Z}\frac{2\pi N}{L}} \hat{f}(k+p)\right| &\leq \frac{L}{N\sqrt{12}}\sqrt{\sum_{k\neq 0, k\in\mathbb{Z}\frac{2\pi N}{L}} |\widehat{f'}(k+p)|^2} \\
&\leq \frac{L}{N\sqrt{12}}\sqrt{\sum_{k\neq 0, k\in\mathbb{Z}\frac{2\pi N}{L}} \left|\frac{\widehat{f^{(m+1)}}(k+p)}{(k+p)^m}\right|^2} \\
&\leq \frac{LM_{m+1}}{N\sqrt{12}}\sqrt{\sum_{k\neq 0, k\in\mathbb{Z}\frac{2\pi N}{L}} \frac{1}{k^{2m}}} \\
&= \frac{LM_{m+1}}{N\sqrt{12}}\sqrt{\sum_{\ell\neq 0, \ell\in\mathbb{Z}} \left(\frac{L}{\ell 2\pi N}\right)^{2m}} \\
&= \frac{L^{m+1}M_{m+1}}{N^{m+1}(2\pi)^m\sqrt{12}}\sqrt{\sum_{\ell\neq 0, \ell\in\mathbb{Z}} \frac{1}{\ell^{2m}}} \\
&= \frac{L^{m+1}M_{m+1}}{N^{m+1}(2\pi)^m}\sqrt{2\sum_{\ell=1}^{\infty} \frac{1}{\ell^{2m}}}.
\end{aligned}
$$

The sum inside the square root is the Riemann zeta function at positive even integers, which is well-known to be

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell^{2m}} = \zeta(2m) = \frac{2^{2m-1}|B(2m)|\pi^{2m}}{(2m)!}$$

where $B(2m)$ is the $2m$th Bernoulli number. The first few values are [**Mat**] $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, and $\zeta(8) = \pi^8/9450$. Detailed discussion of the Riemann zeta function is beyond the scope of this paper; for the present purpose, it suffices to know that, for even positive integers $2m$, $\zeta(2m)$ is bounded above by π^{2m}. Also, $\sqrt{12} < 4$. Thus

$$\sqrt{2\sum_{\ell=1}^{\infty}\frac{1}{\ell^{2m}}} \leq \sqrt{\frac{2^{2m}\pi^{4m}}{(2m)!}} = 2^m\pi^{2m}\sqrt{\frac{1}{(2m)!}}$$

Thus an error bound in the case f is $m+1$ times differentiable is

$$\left(\frac{L}{N}\right)^{m+1}\frac{M_{m+1}2^m\pi^{2m}}{4(2\pi)^m\sqrt{(2m)!}} = \left(\frac{L\pi}{N}\right)^{m+1}\frac{M_{m+1}}{4\pi\sqrt{(2m)!}}.$$

$\square$

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