Induction and the sum of consecutive squares

John Kerl · Math 110, section 2 · Spring 2006

In chapter 5 we encountered formulas for the sum of consecutive integers and the sum of consecutive squares:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

For example,

$$1+2+3+4+5 = \frac{5 \cdot 6}{2} = 15 \qquad \text{and} \qquad 1+4+9+16+25 = \frac{5 \cdot 6 \cdot 11}{6} = 55.$$

But this story has a significant omission: Why are these true — are we just supposed to accept these formulas on faith? Does trying the formula out for a few values of n convince you it's true for *every* n? This question is answered easily enough, using a technique called *induction*. This technique could be taught in college algebra (maybe at some universities it is), and it's useful throughout mathematics.

1 Induction

The idea of an inductive proof is as follows: Suppose you want to show that something is true for all positive integers n. (The catch: you have to already know what you want to prove — induction can prove a formula is true, but it won't produce a formula you haven't already guessed at.)

- **Step 0.** Come up with a formula, and give yourself reason to think it's true.
- **Step 1** (*base case*). Show the formula holds for $n = 1$.
- **Step 2** (*induction step*). *Suppose* it's true for $n - 1$, and then *show* it's true for n.

This is a leapfrog type of argument. It's kind of like evaluating the terms of a sequence: you have a starting point, and you get the next term by doing something to the previous term. This chains along for as long as you have patience.

I showed in class (and your textbook also shows) why the first formula above, the sum-of-consecutive-integers formula, is true. (It's an arithmetic sum.) So we *could* use induction for that formula, but we don't *need* to. I'll illustrate the induction technique by proving that the sum-of-consecutive-squares formula is true.

2 Sum of consecutive squares

Let's see why

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Step 0. How do we come up with the formula? In this case, it comes from your instructor and/or textbook authors. Does it look like it's true? We can try a few values of n. We did this in class, and above we had

$$1 + 4 + 9 + 16 + 25 = \frac{5 \cdot 6 \cdot 11}{6} = 55.$$

The formula claims that the sum *should* be 55, and when we add up the terms, we see it *is* 55.

Step 1. (*Base case*) Show the formula holds for $n = 1$. This is usually the easy part of an induction proof. Here, this is just

$$\sum_{k=1}^{1} k^2 = 1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1.$$

Step 2. (*Induction step*) *Suppose* it's true for $n-1$, and then *show* it's true for n. For this part, you usually need to do some algebraic manipulation. First we write down the expression for the sum of n consecutive squares:

$$\sum_{k=1}^{n} k^2.$$

We're pretending we don't know that this is $n(n+1)(2n+1)/6$, so we don't give ourselves permission to write that down yet. Our job is to come up with that as a *consequence* of what we *are* sure of, namely, that it's true for $n - 1$.

The trick is that this is just the sum of n numbers, so we can split it up into the sum of the first $n-1$ terms, and the very last term all by itself:

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + \sum_{k=n}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2.$$

Now, we're pretending we *are* confident of what the sum of the first $n-1$ consecutive squares is, so we can write this as:

$$\begin{aligned}\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 &= \frac{(n-1)[(n-1)+1][2(n-1)+1]}{6} + n^2 \\ &= \frac{(n-1)(n)(2n-1)}{6} + n^2.\end{aligned}$$

We need to turn this into what we want, which is

$$\frac{n(n+1)(2n+1)}{6}.$$

The question then reduces to:

$$\frac{(n-1)(n)(2n-1)}{6} + n^2 \stackrel{?}{=} \frac{n(n+1)(2n+1)}{6}.$$

All we need to do is FOIL these out and simplify. Since the right-hand side has one big denominator of 6, maybe we could put the left-hand side into the same form.

The left-hand side is

$$\frac{(n-1)(n)(2n-1)}{6} + n^2 = \frac{(n-1)(n)(2n-1) + 6n^2}{6}$$

which FOILs out to

$$\begin{aligned}\frac{(n^2-n)(2n-1)+6n^2}{6} &= \frac{2n^3-n^2-2n^2+n+6n^2}{6}\\ &= \frac{2n^3+3n^2+n}{6}.\end{aligned}$$

The right-hand side FOILs out to

$$\begin{aligned}\frac{n(n+1)(2n+1)}{6} &= \frac{(n^2+n)(2n+1)}{6}\\ &= \frac{2n^3+n^2+2n^2+n}{6}\\ &= \frac{2n^3+3n^2+n}{6}\end{aligned}$$

which is the same thing as the left-hand side.

In summary, we showed that the formula is true for $n = 1$. Then we showed that *if* the sum-of-consecutive-squares formula is true for an integer, *then* it's also true for the next integer. Since it's true for $n = 1$ (the base case), it's true for $n = 2$ by the induction step. Since it's true for $n = 2$, it's true for $n = 3$ by the induction step, and so on. Then we are sure that it's true for any n at all.

3 Two questions

Since we have formulas for the sum of consecutive *integers* and the sum of consecutive *squares*, it's natural to wonder: what do you get when you sum up consecutive *cubes*? For example,

$$\begin{aligned}1 &= 1\\ 1+8 &= 9\\ 1+8+27 &= 36\\ &\vdots\end{aligned}$$

Here's another one — the sum of consecutive *odd* integers:

$$\begin{aligned}1 &= 1\\ 1+3 &= 4\\ 1+3+5 &= 9\\ &\vdots\end{aligned}$$

For either of these: Do you see a pattern? Can you guess a formula? Can you convince yourself it's probably true? Can you prove it's true?

4 More information

There's a nice write-up on *Mathematical induction* in Wikipedia, at `http://en.wikipedia.org`.