The Discrete Berezin Integral

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Abstract

Berezin integration is a concept of mathematical physics which is currently lacking in rigorous definition. In an attempt to remedy this situation, we present a meaningful one-dimensional, discretized version, then characterize it in terms of Fock representations. In future work, this alternative characterization may be useful for returning to continuous, multidimensional constructions. The main topic is preceded by careful exposition of supporting terms and concepts.

Contents

1 Overview

The Berezin integral is a concept arising in mathematical physics. The term “integral” is a misnomer; it is a functional on functions of Grassmann variables. Such “integrals” are not measure-theoretic, and are not defined for indefinite or infinite integrals ([**Rab**]). Here we begin, as guided by [**Pal1**], to attempt to rigorize the Berezin integral via the following scheme:

- Restrict the domain from $\mathbb{R}^3$ to $\mathbb{R}$.
- Discretize the domain from $\mathbb{R}$ to $\mathbb{Z}$.
- Replace continuous operators with range-one finite-difference operators.
- Truncate the domain from $\mathbb{Z}$ to a finite, contiguous subset thereof.
- Now working in the well-understood context of an exterior algebra over a finite-dimensional vector space, re-characterize the Berezin integral using Fock representations.
- In future work, attempt to recover the continuum limit and multiple dimensions in a meaningful way.

This is a work in progress; not all proofs are complete as of this writing.

An outline of this paper is as follows:

- A brief, motivating discussion of physics terminology in section 2.
- Notational conventions in section 3.
- Algebraic preliminaries in section 4.
- Motivation regarding linear operators in section 5.
- Discretization of the domain in section 6.
- Discretized operators in section 7.
- The discrete Berezin integral in section 8.
- The Fock characterization of the discrete Berezin integral in section 9.
- An explicit computation for a two-dimensional, constant-coefficient case is done in section 10.
- Directions for further research are sketched in section 11.

2 Physical background

The discussion in this section serves to motivate concepts, at the basic level needed for this paper. For detailed information, see [**Ber**], [**FS**], and [**Fra**].

2.1 Probablity functions

Quantum mechanics deals with elementary particles not as localized points, but rather as *probability distributions*. Consider a function $\psi : \mathbb{R}^3 \to \mathbb{C}$ which is called the **wave function** associated with a particle. This is taken to mean that

$$\int_D \psi(x)\overline{\psi}(x)\, dx = \int_D |\psi(x)|^2\, dx$$

is the *probability* of finding a particle in the region D of three-space. (For example, if a particle is confined to a line, then $\int_a^b |\psi(x)|^2\, dx$ is the probability of finding the particle between points a and b.) Only the **probability function** $|\psi(x)|^2$ is observable; the complex phase is not observable. Why do we create a mathematical model which uses unobservables? The answer is that the underlying function $\psi(x)$ has linearity (superposition) properties which the probability function does not.

Now, it is certain that the particle is *somewhere* in space. Thus, we require

$$\int_{\mathbb{R}^3} |\psi(x)|^2\, dx = 1.$$

That is, wave functions must be *square integrable* and we have

$$\psi \in L^2(\mathbb{R}^3).$$

In fact, historically such functions lead to the abstract notion of Hilbert space.

Wave functions as presented thus far encode only the probability of a particle being found in a given location: the target space, $\mathbb{C}$, is one-dimensional. Wave functions may also be constructed which encode not only position but also *momentum*, a quantity called *spin*, etc. Here we have functions with values in $\mathbb{C}^N$ for some N. In this case a wave function for a particle is vector-valued, of the form

$$\psi(x) = (\psi_1(x), \ldots, \psi_N(x)).$$

2.2 Fermions and bosons

A system with two particles is described by a single function $\psi(x_1, x_2)$. There exist in nature particles called **bosons** which have **symmetric** wave functions. That is, for all $x_1, x_2 \in \mathbb{R}^3$,

$$\psi(x_1, x_2) = \psi(x_2, x_1).$$

Bosons include photons and gluons; these may be thought of as force carriers. Likewise, there exist particles called **fermions** which have **skew-symmetric** wave functions: for all $x_1, x_2 \in \mathbb{R}^3$,

$$\psi(x_1, x_2) = -\psi(x_2, x_1).$$

Fermions include quarks and electrons; these may be thought of as the stuff of matter. (In fact, certain combinations of matter particles behave as bosons. I am glossing this fact for simplicity of presentation.) Note that this antisymmetry forces $\psi = 0$ at $x_1 = x_2$. This is the *Pauli exclusion principle* which states that fermions do not occupy the same location. This repulsion may be thought of as comprising the "stiffness" of matter. When two automobiles approach each other head-on, their headlight beams pass through one another but the vehicles themselves do not.

A system with n particles is described by a function $\psi(x_1, \ldots, x_n)$; skew-symmetry or symmetry of the particles is indicated by sign change, or lack thereof, respectively, upon transposition of arguments.

2.3 Operators

In physics, extensive use is made of **operators** including the differentiation operator. In this paper, differentiation operators are briefly treated in section 5; the main treatment on Berezin integrals deals with an abstract skew-symmetric operator.

See the above-cited references for more details. Also see [**Lax**] for information on operators on function spaces.

2.4 Creation and annihilation operators

An n-particle state is represented by a wave function

$$\psi(x_1, \ldots, x_n);$$

an $n+1$-particle state is represented by a wave function

$$\phi(x_1, \ldots, x_{n+1}).$$

Operators which map from the former to the latter, or vice versa, are called **creation** and **annihilation operators** respectively. This topic is treated beautifully in chapter 1 of [**Ber**]; discrete versions are presented in section 4.6 of this paper.

2.5 Continuous Berezin integration

We attempt to define expressions of the form

$$\int F(\psi) e^{\mathcal{L}} \prod_{x,j} d\psi_j(x)$$

where

$$\mathcal{L} = \int L\psi(x) dx$$

is an **action functional**. The $\mathcal{L}$ integral is standard; it is the outer integral which is the Berezin integral. Here, $F(\psi)$ is a polynomial in $\psi_1, \ldots, \psi_N$, and L is an abstract skew-symmetric operator. For the purposes of this paper, F is ignored. Thus we want to integrate

$$\int e^{\mathcal{L}} \prod_{x,j} d\psi_j(x).$$

The rules for such integrals ([**Pal1**]) are

$$\int \psi_j \, d\psi_j = 1 \qquad \text{and} \qquad \int 1 \, d\psi_j = 0$$

with antisymmetric extension. A more detailed description is provided in [**Ber**]; see in particular the discussion of *continual integrals* on p. 38.

3 Notation and conventions

Y is used for a general finite-dimensional complex vector space; V is used for discrete bundles as discussed in section 6.

Given a set S, 2^S denotes the set of all subsets of S. Recall that if S is a set containing N elements, then $\#(2^S) = 2^N$ which justifies the notation.

Given a complex vector space Y, write Y^* for the **dual space** of Y, namely, the linear maps from Y to $\mathbb{C}$.

The **span** of a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is written

$$\langle\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}\rangle.$$

The **tensor algebra** on Y is written $\mathcal{T}(Y)$:

$$\mathcal{T}(Y) = \mathbb{C} \oplus Y \oplus (Y \otimes Y) \oplus (Y \otimes Y \otimes Y) \oplus \ldots.$$

4 Linear and multilinear algebra

4.1 Bilinear forms, sesquilinear forms, and conjugations

Throughout this section, let Y be a complex vector space of dimension N.

Definition 4.1. A **sesquilinear form** is a map

$$B : Y \times Y \to \mathbb{C},$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in Y$ and all $a, b \in \mathbb{C}$,

$$\begin{aligned} B(a\mathbf{u}, \mathbf{v}) &= \overline{a}B(\mathbf{u}, \mathbf{v}), \\ B(\mathbf{u}, b\mathbf{v}) &= bB(\mathbf{u}, \mathbf{v}), \\ B(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}) &= B(\mathbf{u}_1, \mathbf{v}) + B(\mathbf{u}_2, \mathbf{v}), \text{ and} \\ B(\mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2) &= B(\mathbf{u}, \mathbf{v}_1) + B(\mathbf{u}, \mathbf{v}_2). \end{aligned}$$

Furthermore, a **symmetric sesquilinear form** is a bilinear form such that for all $\mathbf{u}, \mathbf{v} \in Y$,

$$B(\mathbf{u}, \mathbf{v}) = \overline{B(\mathbf{v}, \mathbf{u})}.$$

Definition 4.2. A **bilinear form** is a map

$$B : Y \times Y \to \mathbb{C},$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in Y$ and all $a, b \in \mathbb{C}$,

$$\begin{aligned} B(a\mathbf{u}, \mathbf{v}) &= aB(\mathbf{u}, \mathbf{v}), \\ B(\mathbf{u}, b\mathbf{v}) &= bB(\mathbf{u}, \mathbf{v}), \\ B(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}) &= B(\mathbf{u}_1, \mathbf{v}) + B(\mathbf{u}_2, \mathbf{v}), \text{ and} \\ B(\mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2) &= B(\mathbf{u}, \mathbf{v}_1) + B(\mathbf{u}, \mathbf{v}_2). \end{aligned}$$

Furthermore, a **symmetric bilinear form** is a bilinear form such that for all $\mathbf{u}, \mathbf{v} \in Y$,

$$B(\mathbf{u}, \mathbf{v}) = B(\mathbf{v}, \mathbf{u}).$$

A **skew-symmetric bilinear form** is a bilinear form such that that for all $\mathbf{u}, \mathbf{v} \in Y$,

$$B(\mathbf{u}, \mathbf{v}) = -B(\mathbf{v}, \mathbf{u}).$$

Definition 4.3. Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ a basis for Y. Then B is said to be **orthonormal** with respect to a symmetric sesquilinear or bilinear form F if $F(\mathbf{b}_i, \mathbf{b}_j) = \delta_{ij}$ for all $i, j = 1, \ldots, N$.

Definition 4.4. A sesquilinear or bilinear form F is said to be **non-degenerate** if for all $\mathbf{v} \in Y$, the kernel of $F(\cdot, \mathbf{v})$ is the zero vector.

Example 4.5. $\triangleright$ Given an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$, one may write a symmetric sesquilinear form $\langle \cdot, \cdot \rangle$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{j=1}^{N} \overline{u}_i v_i$$

or a symmetric bilinear form $(\cdot, \cdot)$

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{N} u_i v_i.$$

The following proposition shows that the converse is true. $\triangleleft$

Proposition 4.6. *For any symmetric non-degenerate sesquilinear or bilinear form F there is a basis for Y that is orthonormal with respect to F*

Proof. Modify Gram-Scmidt. $\square$

There is another, equivalent notion of non-degeneracy: namely, a representation theorem for the dual space.

Proposition 4.7. *A bilinear form B is non-degenerate iff for all $\lambda \in Y^*$, there exists a fixed $\mathbf{u} \in Y$ such that for all $\mathbf{v} \in Y$, $\lambda(v) = \mathbf{u}^T \mathbf{v}$.*

Proof. TBD. $\square$

Definition 4.8. A **conjugation** on Y is a linear transformation $K : Y \to Y$ such that $K^2 = 1$, $K \neq 1$, and for all $\mathbf{v} \in Y$ and all $a \in \mathbb{C}$, $K(a\mathbf{v}) = \overline{a} K(\mathbf{v})$.

Example 4.9. $\triangleright$ Let Y be N-dimensional. If $\mathbf{v} = (v_1, \ldots, v_N)$ in coordinates with respect to some basis, then put $K(\mathbf{v}) = (\overline{v_1}, \ldots, \overline{v_N})$. $\triangleleft$

Definition 4.10. An **orthogonal space** is a complex vector space Y with a distinguished non-degenerate complex bilinear form. We write that form simply as $(\cdot, \cdot)$.

4.2 Pairing and transpose

Definition 4.11. Let Y and Z be vector spaces over the same base field. A **pairing** is a bilinear map from $Y \times Z$ to the base field, written $\langle y, z \rangle$, for $y \in Y$ and $z \in Z$.

Definition 4.12. A pairing $\langle \cdot, \cdot \rangle$ of Y and Z is said to be **non-degenerate** if for all $0 \neq y \in Y$ and all $0 \neq z \in Z$, the linear maps $\langle y, \cdot \rangle : Z \to Z$ and $\langle \cdot, z \rangle : Y \to Y$ have trivial kernels.

Pairings generalize the notions of inner product, integral, and duality of vector spaces.

Example 4.13. $\triangleright$ Any bilinear form on Y is a pairing of Y with itself. In particular, given a function space Y, we write

$$\langle f(x), g(x) \rangle = \int f(x)\, g(x)\, dx.$$

$\triangleleft$

Example 4.14. $\triangleright$ One may pair a vector space Y with its dual Y^*. Then for $\lambda \in Y^*$ and $\mathbf{v} \in Y$ we write

$$\langle \lambda, \mathbf{v} \rangle = \lambda(\mathbf{v}).$$

$\triangleleft$

Definition 4.15. Let $A : Y_1 \to Y_2$ be a linear transformation of vector spaces. Further suppose that we have non-degenerate pairings of Y_1 with Z_1, and of Y_2 with Z_2. Then the **transpose** of A is the linear transformation $A^\tau : Z_2 \to Z_1$ such that

$$\langle y_2, A z_1 \rangle = \langle A^\tau y_2, z_1 \rangle$$

for all $y_2 \in Y_2$ and $z_1 \in Z_2$. A diagram is as follows:

$$\begin{array}{ccc} Y_1 & \xrightarrow{A} & Y_2 \\ \langle \cdot, \cdot \rangle & & \langle \cdot, \cdot \rangle \\ Z_1 & \xleftarrow{A^\tau} & Z_2 \end{array}$$

Remark 4.16. Using an basis which is orthonormal with respect to the form, the transpose may be computed easily by taking the usual matrix transpose.

4.3 Quadratic forms

Throughout this section, let Y be a complex vector space.

Definition 4.17. A **quadratic form** is a map

$$Q : Y \to \mathbb{C}$$

such that for all $\mathbf{u}, \mathbf{v} \in Y$ and all $a \in \mathbb{C}$,

$$Q(a\mathbf{u}) = a^2 Q(\mathbf{u}),$$

and such that the map

$$B : Y \times Y \to \mathbb{C}$$

given by

$$B(\mathbf{u}, \mathbf{v}) = Q(\mathbf{u} + \mathbf{v}) - Q(\mathbf{u}) - Q(\mathbf{v})$$

is bilinear. This map B is called the **associated bilinear form** B for Q. (Note that B is symmetric by construction.)

Remark 4.18. The association may be done the other way as well: given a symmetric bilinear form B, we may create a quadratic form Q via

$$Q(\mathbf{u}) = \frac{1}{2} B(\mathbf{u}, \mathbf{u}).$$

Proposition 4.19. *Given B, the associated Q is in fact bilinear.*

Proof. Let $a \in \mathbb{C}$ and $\mathbf{u} \in Y$. Then

$$Q(a\mathbf{u}) = B(a\mathbf{u}, a\mathbf{u}) = a^2 B(\mathbf{u}, \mathbf{u}) = a^2 Q(\mathbf{u}).$$

For the second axiom, it suffices to show that, given B and the associated quadratic form Q, the bilinear form B' associated to Q is again B. We have

$$\begin{aligned}
B'(\mathbf{u}, \mathbf{v}) &= Q(\mathbf{u} + \mathbf{v}) - Q(\mathbf{u}) - Q(\mathbf{v}) \\
&= \frac{1}{2} \left[B(\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) - B(\mathbf{u}, \mathbf{u}) - B(\mathbf{v}, \mathbf{v}) \right] \\
&= \frac{1}{2} \left[B(\mathbf{u}, \mathbf{u}) + B(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, \mathbf{u}) + B(\mathbf{v}, \mathbf{v}) - B(\mathbf{u}, \mathbf{u}) - B(\mathbf{v}, \mathbf{v}) \right] \\
&= \frac{1}{2} \left[B(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, \mathbf{u}) \right] \\
&= B(\mathbf{u}, \mathbf{v})
\end{aligned}$$

since B is symmetric. $\square$

4.4 Exterior algebras

I take exterior algebras to be known to the reader. One may consult any graduate algebra text, e.g. [**DF**] or [**Rot**], or any graudate differential geometry text such as [**Lee2**]. Here, I make note of my conventions and notation. (Caution is due: in the literature, tensor algebras harbor an annoying diversity of notation.)

Definition 4.20. Recall $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$, i.e. $\mathbf{u} \wedge \mathbf{v} + \mathbf{v} \wedge \mathbf{u} = 0$, and

$$\Lambda(Y) = \mathcal{T}(Y) / \langle \mathbf{uv} + \mathbf{vu} \rangle.$$

Definition 4.21. Following [**Pal2**], we define the **scale factor** for the wedge product of a k-vector and an ℓ-vector as

$$\sqrt{\frac{(k+\ell)!}{k!|\ell!}}.$$

This choice is perhaps non-standard, but is important as will be seen below.

Given a complex vector space Y, write $\Lambda^0(Y) = \mathbb{C}$, $\Lambda^1(Y) = Y$, $\Lambda^2(Y)$ for alternating 2-tensors on Y, etc. Then write

$$\Lambda(Y) = \mathbb{C} \oplus Y \oplus \Lambda^2(Y) \oplus \ldots.$$

for the graded alternating tensor algebra which is endowed with the wedge product.

Remark 4.22. Recall that if $\mathbf{u} = (u_1, \ldots, u_N) \in Y$ and $\mathbf{v} = (v_1, \ldots, v_N) \in Y$ then the **tensor product** may be visualized in coordinates as an **outer product**

$$(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j,$$

i.e. the two-dimensional $N \times N$ array

$$\begin{pmatrix} u_1v_1 & \cdots & u_1v_N \\ \vdots & & \vdots \\ u_Nv_1 & \cdots & u_Nv_N \end{pmatrix}.$$

This may be computed by hand using the tableau

$$\begin{array}{c|ccc} & v_1 & \ldots & v_N \\ \hline u_1 & u_1v_1 & \cdots & u_1v_N \\ \vdots & \vdots & & \vdots \\ u_N & u_Nv_1 & \cdots & u_Nv_N. \end{array}$$

Furthermore, since

$$\mathbf{u} \wedge \mathbf{v} = \frac{1}{\sqrt{2}} \left(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} \right),$$

we have in coordinates

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} &= \frac{1}{\sqrt{2}} \begin{pmatrix} u_1v_1 & \cdots & u_1v_N \\ \vdots & & \vdots \\ u_Nv_1 & \cdots & u_Nv_N \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} u_1v_1 & \cdots & u_Nv_1 \\ \vdots & & \vdots \\ u_1v_N & \cdots & u_Nv_N \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} u_1v_1 - u_1v_1 & \cdots & u_1v_N - u_Nv_1 \\ \vdots & & \vdots \\ u_Nv_1 - u_1v_N & \cdots & u_Nv_N - u_Nv_N \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_1v_2 - u_2v_1 & \cdots & u_1v_{N-1} - u_{N-1}v_1 & u_1v_N - u_Nv_1 \\ u_2v_1 - u_1v_2 & 0 & & & \vdots \\ \vdots & & \ddots & & \\ u_{N-1}v_1 - u_1v_{N-1} & & & 0 & u_{N-1}v_N - u_Nv_{N-1} \\ u_Nv_1 - u_1v_N & \cdots & & u_Nv_{N-1} - u_{N-1}v_N & 0 \end{pmatrix}. \end{aligned}$$

Remark 4.23. In particular, the wedge of two standard-basis vectors contains a single 1, a single -1, and the rest zeroes. For example, with $N = 4$,

$$\mathbf{e}_3 \wedge \mathbf{e}_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Remark 4.24. A bi-vector as described in remark 4.22 has N^2 components when written in coordinates. Similarly, a tri-vector, i.e. an element of $\Lambda^3(Y)$, has N^3 components, and in general a k-vector (an element of $\Lambda^k(Y)$) has N^k components. In particular, a top-level form (an element of $\Lambda^N(Y)$) has N^N components. Recall, however, that the dimension of $\Lambda^k(Y)$ is $\binom{N}{k}$ and in particular $\Lambda^N(Y)$ is one-dimensional. Thus, naïve calculations such as those in remark 4.22 produce an overabundance of coefficients; a more elegant computation extracts a single scalar from a top-level form.

Example 4.25. $\triangleright$ Let q be an arbitrary bivector on a 4-dimensional space:

$$q = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

To compute $q \wedge q$ from the definition, one could compute $q \otimes q$ and then compute the alternation of the result. For the former we have (disregarding the scale factor):

$$q \otimes q = \left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & a^2 & ab & ac & 0 & ab & b^2 & bc & 0 & ac & bc & c^2 \\
0 & 0 & 0 & 0 & -a^2 & 0 & ad & ae & -ab & 0 & bd & be & -ac & 0 & cd & ce \\
0 & 0 & 0 & 0 & -ab & -ad & 0 & af & -b^2 & -bd & 0 & bf & -bc & -cd & 0 & cf \\
0 & 0 & 0 & 0 & -ac & -ae & -af & 0 & -bc & -be & -bf & 0 & -c^2 & -ce & -cf & 0 \\
\hline
0 & -a^2 & -ab & -ac & 0 & 0 & 0 & 0 & 0 & ad & bd & cd & 0 & ae & be & ce \\
a^2 & 0 & -ad & -ae & 0 & 0 & 0 & 0 & -ad & 0 & d^2 & de & -ae & 0 & de & e^2 \\
ab & ad & 0 & -af & 0 & 0 & 0 & 0 & -bd & -d^2 & 0 & df & -be & -de & 0 & ef \\
ac & ae & af & 0 & 0 & 0 & 0 & 0 & -cd & -de & -df & 0 & -ce & -e^2 & -ef & 0 \\
\hline
0 & -ab & -b^2 & -bc & 0 & -ad & -bd & -cd & 0 & 0 & 0 & 0 & 0 & af & bf & cf \\
ab & 0 & -bd & -be & ad & 0 & -d^2 & -ed & 0 & 0 & 0 & 0 & -af & 0 & df & ef \\
b^2 & bd & 0 & -bf & bd & d^2 & 0 & -fd & 0 & 0 & 0 & 0 & -bf & -df & 0 & f^2 \\
bc & be & bf & 0 & cd & de & df & 0 & 0 & 0 & 0 & 0 & -cf & -ef & -f^2 & 0 \\
\hline
0 & -ac & -bc & -c^2 & 0 & -ae & -be & -ce & 0 & -af & -bf & -cf & 0 & 0 & 0 & 0 \\
ac & 0 & -dc & -ce & ae & 0 & -de & -e^2 & af & 0 & -df & -ef & 0 & 0 & 0 & 0 \\
bc & cd & 0 & -cf & be & de & 0 & -ef & bf & df & 0 & -f^2 & 0 & 0 & 0 & 0 \\
c^2 & ce & cf & 0 & ce & e^2 & ef & 0 & cf & ef & f^2 & 0 & 0 & 0 & 0 & 0
\end{array} \right]$$

However, this is beginning to get messy: there are 256 elements in this array, and 24 elements in $\mathcal{S}_4$. The point of this example is to convince the reader of the benefits of alternative techniques: not all 256 elements are independent, and a more efficient approach is possible. Note from remark 4.23 that q is simply

$$\begin{aligned} q \quad = \quad a\,\mathbf{e}_1 \wedge \mathbf{e}_2 \quad &+ \quad b\,\mathbf{e}_1 \wedge \mathbf{e}_3 \quad + \quad c\,\mathbf{e}_1 \wedge \mathbf{e}_4 \\ &+ \quad d\,\mathbf{e}_2 \wedge \mathbf{e}_3 \quad + \quad e\,\mathbf{e}_2 \wedge \mathbf{e}_4 \\ &\qquad\qquad\qquad\quad + \quad f\,\mathbf{e}_3 \wedge \mathbf{e}_4. \end{aligned}$$

Then $q \wedge q$ is obtained by squaring this sum. Note that any term in the product which contains a repeated

$\mathbf{e}_i$ will be zero, so we have only

$$\begin{aligned}
q \wedge q &= af\,\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 + be\,\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4 + cd\,\mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\
&+ dc\,\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4 + eb\,\mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_3 + fa\,\mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \\
&= 2af\,\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 + 2be\,\mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4 + 2cd\,\mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \\
&= 2(af - be + cd)\,\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4.
\end{aligned}$$

It may be shown that $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ expands such that

$$q \wedge q = 2(af - be + cd) \left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

Since 4-vectors are top-level for $N = 4$, $\Lambda^4(Y)$ is 1-dimensional and spanned by $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$. Thus, all information about the 256 elements of $q \wedge q$ is encapsulated in the single scalar $2(af - be + cd)$. $\triangleleft$

Remark 4.26. For $\phi \in Y^*$ and $v \in Y$ we have the pairing

$$\langle \phi, v \rangle = \phi(v).$$

Recall that $Y^* = \Lambda^1(Y^*)$ and $Y = \Lambda^1(Y)$. Moreover, for $\Phi = \phi_1 \wedge \cdots \wedge \phi_k \in \Lambda^k(Y^*)$ and $\mathbf{v} = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(Y)$ we have the pairing

$$\langle \Phi, \mathbf{v} \rangle = \langle (\phi_1 \wedge \cdots \wedge \phi_k), (v_1 \wedge \cdots \wedge v_k) \rangle = \begin{vmatrix} \phi_1(v_1) & \cdots & \phi_1(v_k) \\ \vdots & & \vdots \\ \phi_k(v_1) & \cdots & \phi_k(v_k) \end{vmatrix}.$$

The wedge scale factor in definition 4.21 is chosen precisely to make this statement work. Details are probably in [**Pal2**]; if not (or even if so), I should include them here.

4.5 Clifford algebras

The geometric origin of Clifford and Grassmann algebras is presented beautifully in sections 11.5 and 11.6 of [**Pen**]. As well, the skew-symmetry of Clifford algebras is used to mimic the skew-symmetry of fermions; see [**Ber**].

Definition 4.27. Let Y be a complex vector space, and let S be a symmetric bilinear form on Y. (That is, Y is an orthogonal space in the sense of definition 4.10.) The **Clifford algebra** of Y and S, written $\mathrm{Cliff}(Y)$ without explicit mention of S, is the tensor algebra on Y modulo the **anticommutation relations** $\mathbf{uv} + \mathbf{vu} = S(\mathbf{u}, \mathbf{v})$. That is,

$$\mathrm{Cliff}(Y) = T(Y) \big/ \langle \mathbf{uv} + \mathbf{vu} - S(\mathbf{uv}) \rangle$$

Note that the trivial (and highly degenerate) form $S(\mathbf{u}, \mathbf{v}) = 0$ is symmetric and bilinear. The Clifford algebra with this form is precisely the exterior algebra (definition 4.20). Thus, Clifford algebras generalize the concept of exterior algebra. However, we usually take the symmetric bilinear forms for our Clifford algebras to be non-degenerate.

In the exterior algebra $\Lambda(Y)$, we compute with wedge products of vectors and make use of the anticommutation relation $\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}$; we rarely need to mention a basis for Y unless, say, we are discussing the dimension of $\Lambda(Y)$ as a vector space. Computation in $\mathrm{Cliff}(Y)$ is a bit more subtle. Definition 4.27 tells us how to formally juxtapose two vectors when we take their product. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ be an orthonormal basis for Y with respect to S, i.e. $S(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$. Then the anticommutation relation specializes to

$$\mathbf{e}_j \mathbf{e}_i = -\mathbf{e}_i \mathbf{e}_j, \ i \neq j$$

(i.e. for orthonormal vectors, we still have simple anticommutativity as in the exterior algebra) and

$$\mathbf{e}_i \mathbf{e}_i = 1/2.$$

In fact, it is clear that with respect to S,

$$\mathbf{u} \perp \mathbf{v} \iff \mathbf{uv} = -\mathbf{vu}.$$

It may be shown (but is not shown here) that the Clifford algebra has the same dimension as the exterior algebra, namely, 2^N where N is the dimension of Y.

Clifford algebras play a key role in Fock representations (section 4.11).

4.6 Creation and annihilation operators

In this section, let Y be a finite-dimensional complex vector space. We present a discretized version of the creation and annihilation operators which were briefly mentioned in section 2.4.

Definition 4.28. For $u \in Y$ and $\mathbf{v} = v_1 \wedge \cdots \wedge v_{k-1} \in \Lambda^{k-1}(Y)$, define the **creation operator**

$$c(u) : \Lambda^{k-1}(Y) \to \Lambda^k(Y),$$

via

$$c(u)\mathbf{v} = u \wedge \mathbf{v}.$$

By varying k we have, on the full alternating tensor algebra, $c(u) : \Lambda(Y) \to \Lambda(Y)$. This means that also, for $\lambda \in Y^*$ and $\Phi = \phi_1 \wedge \cdots \wedge \phi_{k-1} \in \Lambda^{k-1}(Y^*)$, we have a creation operator

$$c(\lambda) : \Lambda^{k-1}(Y^*) \to \Lambda^k(Y^*)$$

(and hence $c(\lambda) : \Lambda(Y^*) \to \Lambda(Y^*)$) via

$$c(\lambda)\Phi = \lambda \wedge \Phi.$$

Example 4.29. $\triangleright$ We have

$$c(\mathbf{e}_1)\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

and

$$c(\mathbf{e}_1)\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 = 0.$$

Recall however that $0 \in \Lambda^3(Y)$, so the definition is intact. $\triangleleft$

Definition 4.30. Formally, the **annihilation operator** is the transpose of the creation operator. Concretely, let

$$\lambda \in Y^*,$$

let

$$\Phi = \phi_1 \wedge \cdots \wedge \phi_{k-1} \in \Lambda^{k-1}(Y^*),$$

and let

$$\mathbf{v} = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(Y).$$

Recall from definition 4.28 that

$$c(\lambda) : \Lambda^{k-1}(Y^*) \to \Lambda^k(Y^*)$$

and recall from remark 4.26 that we have pairings of $\Lambda^{k-1}(Y^*)$ with $\Lambda^{k-1}(Y)$ and of $\Lambda^k(Y^*)$ with $\Lambda^k(Y)$. From the definition of transpose (definition 4.15), we have

$$a(\lambda) = c(\lambda)^\tau : \Lambda^k(Y) \to \Lambda^{k-1}(Y),$$

specified by

$$\langle c(\lambda)\Phi, \mathbf{v}\rangle = \langle \Phi, a(\lambda)\mathbf{v}\rangle.$$

Also note the following diagram:

$$\begin{array}{ccc} \Lambda^{k-1}(Y^*) & \overset{c(\lambda)}{\longrightarrow} & \Lambda^k(Y^*) \\ \langle\cdot,\cdot\rangle & & \langle\cdot,\cdot\rangle \\ \Lambda^{k-1}(Y) & \overset{c(\lambda)^\tau = a(\lambda)}{\longleftarrow} & \Lambda^k(Y) \end{array}.$$

Definition 4.31. Let $\mathbf{v} = v_1 \wedge \cdots \wedge v_k$. Write

$$\hat{\mathbf{v}}_j = v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k.$$

That is, $\hat{\mathbf{v}}_j$ omits v_j from $\mathbf{v}$. Likewise,

$$\hat{\mathbf{v}}_{ij} = v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k,$$

omitting v_i and v_j from $\mathbf{v}$.

Proposition 4.32. *The annihilation operator is given by the explicit formula:*

$$a(\lambda)\mathbf{v} \quad = \quad \sum_{j=1}^{k} (-1)^{j-1} \lambda(v_j)\hat{\mathbf{v}}_j.$$

Proof. Consider the $k \times k$ determinant for wedge pairings, as discussed in remark 4.26:

$$\begin{aligned} \langle c(\lambda)\Phi, \mathbf{v}\rangle &= \langle \Phi, a(\lambda)\mathbf{v}\rangle \\ \langle c(\lambda)\Phi, \mathbf{v}\rangle &= \langle \lambda \wedge \phi_1 \wedge \cdots \wedge \phi_k, v_1 \wedge \cdots \wedge v_k\rangle \\ &= \begin{vmatrix} \lambda(v_1) & \cdots & \lambda(v_k) \\ \phi_1(v_1) & \cdots & \phi_1(v_k) \\ \vdots & & \vdots \\ \phi_k(v_1) & \cdots & \phi_k(v_k) \end{vmatrix}. \end{aligned}$$

Now we simply expand along the top row, noting that the jth minor is the $(k-1)\times(k-1)$ determinant

$$\begin{vmatrix} \phi_1(v_1) & \cdots & \phi_1(v_{j-1}) & \phi_1(v_{j+1}) & \cdots & \phi_1(v_k) \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_k(v_k) & \cdots & \phi_k(v_{j-1}) & \phi_k(v_{j+1}) & \cdots & \phi_k(v_k) \end{vmatrix}.$$

Writing the jth minor as $\langle \Phi, \hat{\mathbf{v}}_j \rangle$, we obtain

$$\langle \Phi, a(\lambda)\mathbf{v} \rangle \quad = \quad \sum_{j=1}^{k} (-1)^{j-1} \lambda(v_j) \langle \Phi, \hat{\mathbf{v}}_j \rangle = \left\langle \Phi, \sum_{j=1}^{k} (-1)^{j-1} \lambda(v_j) \hat{\mathbf{v}}_j \right\rangle.$$

Since this holds for all $\mathbf{v}$, we conclude (note: I need to elaborate here, using non-degeneracy of the pairing) that

$$a(\lambda)\mathbf{v} \quad = \quad \sum_{j=1}^{k} (-1)^{j-1} \lambda(v_j) \hat{\mathbf{v}}_j.$$

□

Proposition 4.33. *The creation and annihilation operators are linear transformations on $\Lambda(Y)$.*

Proof. The linearity of the creation operator is obtained from arithmetic rules in the exterior algebra:

$$c(au)\mathbf{w} = (au) \wedge \mathbf{w} = a(u \wedge \mathbf{w})$$

and

$$c(u+v)\mathbf{w} = (u+v) \wedge \mathbf{w} = u \wedge \mathbf{w} + v \wedge \mathbf{w} = c(u)\mathbf{w} + c(v)\mathbf{w}.$$

The annihilation operator is linear since it is the transpose of a linear operator (see definition 4.15). □

Example 4.34. ▷ Let $\lambda = \mathbf{e}_2^*$ and $\mathbf{v} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Then

$$\begin{aligned} a(\lambda)(\mathbf{v}) &= a(\mathbf{e}_2^*)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \\ &= \mathbf{e}_2^*(\mathbf{e}_1)(\mathbf{e}_2 \wedge \mathbf{e}_3) - \mathbf{e}_2^*(\mathbf{e}_2)(\mathbf{e}_1 \wedge \mathbf{e}_3) + \mathbf{e}_2^*(\mathbf{e}_3)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\ &= -\mathbf{e}_1 \wedge \mathbf{e}_3. \end{aligned}$$

Recall that when we compute determinants, we may omit the alternating signs if we barrel-shift the minors. E.g. we normally write

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - gf) + c(dh - ge)$$

but instead we may write

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) + b(fg - id) + c(dh - ge).$$

Using this fact, we may write

$$\begin{aligned} a(\lambda)(\mathbf{v}) &= a(\mathbf{e}_2^*)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \\ &= \mathbf{e}_2^*(\mathbf{e}_1)(\mathbf{e}_2 \wedge \mathbf{e}_3) + \mathbf{e}_2^*(\mathbf{e}_2)(\mathbf{e}_3 \wedge \mathbf{e}_1) + \mathbf{e}_2^*(\mathbf{e}_3)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\ &= \mathbf{e}_3 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_3. \end{aligned}$$

◁

Example 4.35. $\triangleright$ It will be shown below that the creation and annihilation operators anticommute. The proof of the latter wends through a dense thicket of summation symbols, which will be illuminated by the following concrete example. Let $\lambda, \mu \in Y^*$ and let $\mathbf{w} = w_1 \wedge w_2 \wedge w_3 \wedge w_4$. Then, using the formula from proposition 4.32, we have

$$a(\lambda)a(\mu)\mathbf{w} \quad = \quad \begin{array}{rlllll} \lambda(w_1) & [& & \mu(w_2)w_3 \wedge w_4 & -\mu(w_3)w_2 \wedge w_4 & +\mu(w_4)w_2 \wedge w_3 \quad] \\ -\lambda(w_2) & [& \mu(w_1)w_3 \wedge w_4 & & -\mu(w_3)w_1 \wedge w_4 & +\mu(w_4)w_1 \wedge w_3 \quad] \\ +\lambda(w_3) & [& \mu(w_1)w_2 \wedge w_4 & -\mu(w_2)w_1 \wedge w_4 & & +\mu(w_4)w_1 \wedge w_2 \quad] \\ -\lambda(w_4) & [& \mu(w_1)w_2 \wedge w_3 & -\mu(w_2)w_1 \wedge w_3 & +\mu(w_3)w_1 \wedge w_2 & \quad] \end{array}$$

and

$$a(\mu)a(\lambda)\mathbf{w} \quad = \quad \begin{array}{rlllll} \mu(w_1) & [& & \lambda(w_2)w_3 \wedge w_4 & -\lambda(w_3)w_2 \wedge w_4 & +\lambda(w_4)w_2 \wedge w_3 \quad] \\ -\mu(w_2) & [& \lambda(w_1)w_3 \wedge w_4 & & -\lambda(w_3)w_1 \wedge w_4 & +\lambda(w_4)w_1 \wedge w_3 \quad] \\ +\mu(w_3) & [& \lambda(w_1)w_2 \wedge w_4 & -\lambda(w_2)w_1 \wedge w_4 & & +\lambda(w_4)w_1 \wedge w_2 \quad] \\ -\mu(w_4) & [& \lambda(w_1)w_2 \wedge w_3 & -\lambda(w_2)w_1 \wedge w_3 & +\lambda(w_3)w_1 \wedge w_2 & \quad]. \end{array}$$

Here it is clear that $a(\mu)a(\lambda)\mathbf{w} = -a(\lambda)a(\mu)\mathbf{w}$, term by term. $\triangleleft$

Proposition 4.36. *The creation and annihilation operators satisfy the* ***anticommutation relations***

$$\begin{aligned} c(u)\,c(v) + c(v)\,c(u) &= 0, \\ a(\lambda)\,a(\mu) + a(\mu)\,a(\lambda) &= 0, \\ a(\lambda)\,c(u) + c(u)\,a(\lambda) &= \lambda(u). \end{aligned}$$

Proof. For the creation operator, let $\mathbf{w} = w_1 \wedge \cdots \wedge w_{k-2} \in \Lambda^{k-2}(Y)$. Then

$$c(u)\,c(v)(\mathbf{w}) = u \wedge v \wedge \mathbf{w} = -v \wedge u \wedge \mathbf{w} = -c(v)\,c(u)(\mathbf{w}).$$

For the annihilation operator, we simply generalize example 4.35 using summation notation. Let $\lambda, \mu \in Y^*$, and let $\mathbf{w} = w_1 \wedge \cdots \wedge w_k \in \Lambda^k(Y)$. Then

$$\begin{aligned} a(\lambda)a(\mu)\mathbf{w} &= \sum_{i=1}^{k}(-1)^{i+1}\lambda(w_i)\left[\sum_{j=1}^{i-1}(-1)^{j+1}\mu(w_j)\hat{\mathbf{w}}_{ij} + \sum_{j=i+1}^{k}(-1)^{j}\mu(w_j)\hat{\mathbf{w}}_{ij}\right] \\ &= \sum_{i=1}^{k}\sum_{j=1}^{i-1}(-1)^{i+j}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \quad + \quad \sum_{i=1}^{k}\sum_{j=i+1}^{k}(-1)^{i+j+1}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \\ a(\mu)a(\lambda)\mathbf{w} &= \sum_{j=1}^{k}(-1)^{j+1}\mu(w_j)\left[\sum_{i=1}^{j-1}(-1)^{i+1}\lambda(w_i)\hat{\mathbf{w}}_{ij} + \sum_{i=j+1}^{k}(-1)^{i}\lambda(w_i)\hat{\mathbf{w}}_{ij}\right] \\ &= \sum_{j=1}^{k}\sum_{i=1}^{j-1}(-1)^{i+j}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \quad + \quad \sum_{j=1}^{k}\sum_{i=j+1}^{k}(-1)^{i+j+1}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij}. \end{aligned}$$

Notice that in example 4.35 the diagonal entries are blank, and we may split the double sum into the sum of above-diagonal entries and the sum of below-diagonal entries. Then

$$\begin{aligned}
a(\lambda)a(\mu)\mathbf{w} &= \sum_{i<j}(-1)^{i+j+1}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \quad + \quad \sum_{i>j}(-1)^{i+j}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \\
a(\mu)a(\lambda)\mathbf{w} &= \sum_{j<i}(-1)^{i+j+1}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \quad + \quad \sum_{j>i}(-1)^{i+j}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \\
&= \sum_{i<j}(-1)^{i+j}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \quad + \quad \sum_{i>j}(-1)^{i+j+1}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \\
&= -\sum_{i<j}(-1)^{i+j+1}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \quad - \quad \sum_{i>j}(-1)^{i+j}\lambda(w_i)\mu(w_j)\hat{\mathbf{w}}_{ij} \\
&= -a(\lambda)a(\mu)\mathbf{w}
\end{aligned}$$

which is what was to be shown.

For the anticommutation relation between the creation and annihilation operators, we again use a test k-vector. Let $\mathbf{w} = w_1 \wedge \cdots \wedge w_k \in \Lambda^k(Y)$. Then, using the formula from proposition 4.32, we have

$$\begin{aligned}
a(\lambda)\,c(u)\,\mathbf{w} &= a(\lambda)\ (u \wedge w_1 \wedge \cdots \wedge w_k) \\
&= \lambda(u)\,w_1 \wedge \cdots \wedge w_k + u \wedge \left(\sum_{i=1}^{k}(-1)^i\lambda(w_i)\hat{\mathbf{w}}_i\right) \\
&= \lambda(u)\,w_1 \wedge \cdots \wedge w_k + \sum_{i=1}^{k}(-1)^i\lambda(w_i)u \wedge \hat{\mathbf{w}}_i
\end{aligned}$$

and

$$\begin{aligned}
c(u)\,a(\lambda)\,\mathbf{w} &= u \wedge \left(\sum_{i=1}^{k}(-1)^{i+1}\lambda(w_i)\hat{\mathbf{w}}_i\right) \\
&= \sum_{i=1}^{k}(-1)^{i+1}\lambda(w_i)u \wedge \hat{\mathbf{w}}_i \\
&= -\sum_{i=1}^{k}(-1)^{i}\lambda(w_i)u \wedge \hat{\mathbf{w}}_i
\end{aligned}$$

so that

$$\begin{aligned}
a(\lambda)\,c(u)\,\mathbf{w} + c(u)\,a(\lambda)\,\mathbf{w} &= \lambda(u)\,w_1 \wedge \cdots \wedge w_k \\
&= \lambda(u)\mathbf{w}.
\end{aligned}$$

Since $\mathbf{w}$ was arbitrary, we conclude that

$$a(\lambda)\,c(u) + c(u)\,a(\lambda) = \lambda(u)$$

as functions. □

4.7 Bra-ket notation

In this section we introduct a notation which is commonplace in the physics literature. (See any of the physics references in the appendix for more information.)

Definitions 4.37. Let Y be a complex vector space. Then we write

$$\begin{aligned} Y \ni v &= |\, v\,\rangle \\ Y^* \ni \lambda &= \langle \lambda\,| \\ \langle \lambda, v\rangle &= \langle\, \lambda\,|\, v\,\rangle. \end{aligned}$$

One says that a functional (or covector) λ is a **bra** and a vector v is a **ket**. This notation is collectively referred to as **bra-ket** notation. We also write

$$\Lambda(Y) \ni 1 \oplus 0 \oplus 0 \oplus \cdots \oplus 0 = |\, 0\,\rangle.$$

Note that this notation is a little misleading: $|\,0\,\rangle$ is really 1.

Remark 4.38. Bra-ket notation, along with creation and annihilation operators (section 4.6) gives physicists an alternative way to write elements of exterior algebras. This is often used in place of the grading and wedges which are preferred in the mathematical literature. In particular, for $u, v \in Y$:

$$\begin{aligned} c(u)|\,0\,\rangle &= u \wedge 1 = u \\ c(u)c(v)|\,0\,\rangle &= c(u)(v) = u \wedge v \\ a(u)|\,0\,\rangle &= 0 \\ Au \wedge v &= c(Au)c(v)|\,0\,\rangle. \end{aligned}$$

4.8 Pfaffians

The Pfaffian is a square root of the determinant which is defined on skew-symmetric matrices. It will be used for efficient computation of the discrete Berezin integral in section 8.

Definition 4.39. Let A be an $N \times N$ skew-symmetric matrix, with N even. Define the **associated bivector** b_A to be

$$b_A = \sum_{i<j} A_{ij} \mathbf{e}_i \wedge \mathbf{e}_j.$$

Recall that from remark 4.22 that a bivector, when written in coordinates, is simply a two-dimensional array of numbers. Also recall that bivectors are doubly contravariant with respect to change of coordinates, whereas matrices are of mixed variance. Nonetheless, with respect to a particular coordinate system, the skew-symmetric matrix A and its associated bivector b_A contain precisely the same numbers in their two-dimensional arrays.

Proposition 4.40. *With respect to a particular coordinate system, A and b_A as defined above may be viewed as two-dimensional arrays containing the same elements.*

Proof. Let

$$A = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1,N-1} & A_{1,N} \\ -A_{12} & 0 & & A_{2,N-1} & A_{1,N} \\ \vdots & & \ddots & & \vdots \\ -A_{1,N-1} & & & 0 & A_{N-1,N} \\ -A_{1,N} & & & -A_{N-1,N} & 0 \end{bmatrix}$$

be an arbitrary $N \times N$ matrix. Recall from remark 4.23 that $\mathbf{e}_i \wedge \mathbf{e}_j$ may be viewed as an $N \times N$ array with $+1$ at the i, j element, -1 at the j, i elements, and 0 elsewhere. Summing as in definition 4.39 yields the array

$$b_A = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1,N-1} & A_{1,N} \\ -A_{12} & 0 & & A_{2,N-1} & A_{1,N} \\ \vdots & & \ddots & & \vdots \\ -A_{1,N-1} & & & 0 & A_{N-1,N} \\ -A_{1,N} & & & -A_{N-1,N} & 0 \end{bmatrix}.$$

□

Example 4.41. Consider a general 4×4 skew-symmetric matrix:

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

Recall from remark 4.23 that $\mathbf{e}_i \wedge \mathbf{e}_j$ is a two-dimensional array with (disregarding scale factors) 1 at the i, j element and -1 at the j, i element, and zeroes elsewhere. Thus, using definition 4.39, we have

$$\begin{array}{rcl} b_A & = & a\mathbf{e}_1 \wedge \mathbf{e}_2 + b\,\mathbf{e}_1 \wedge \mathbf{e}_3 + c\,\mathbf{e}_1 \wedge \mathbf{e}_4 \\ & & \quad + d\,\mathbf{e}_2 \wedge \mathbf{e}_3 + e\,\mathbf{e}_2 \wedge \mathbf{e}_4 \\ & & \quad + f\,\mathbf{e}_3 \wedge \mathbf{e}_4 \end{array}$$

$$b_A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

Definition 4.42. Let A be an $N \times N$ skew-symmetric matrix, with N even. Then the **Pfaffian** of A, written $\mathrm{Pf}(A)$, is specified by

$$\frac{1}{(N/2)!}\, b_A^{N/2} = \mathrm{Pf}(A)\, \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_N.$$

Example 4.43. ▷ Use the 4×4 skew-symmetric matrix A from example 4.41. Working directly from definition 4.42, with $N = 4$, we have

$$\frac{1}{2}\, b_A^2 = \mathrm{Pf}(A)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4.$$

The product b_A^2 was computed in example 4.25: it was found to be $2(af - be + cd)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$. Thus, we have

$$\mathrm{Pf}(A) = af - be + cd.$$

Furthermore, a direct computation reveals that

$$\det(A) = (af - be + cd)^2,$$

as expected. ◁

Proposition 4.44. *For any skew-symmetric matrix A,*

$$\mathrm{Pf}^2(A) = \det(A)$$

Proof. TBD. Or cite .... □

Corollary 4.45. $\mathrm{Pf}(\lambda A) = \lambda^{N/2}\mathrm{Pf}(A)$.

Proof. Using the known N-linearity of the determinant, we have

$$\mathrm{Pf}^2(\lambda A) = \det(\lambda A) = \lambda^N \det(A) = \lambda^N \mathrm{Pf}^2(A).$$

$\square$

The computation of Pfaffians for matrices of larger dimensions is facilitated by the following formulas.

Proposition 4.46. *The Pfaffian of an odd-dimensional skew-symmetric matrix is zero.*

Proof. TBD. $\square$

Definition 4.47. Let A be an $N \times N$ matrix. Define $\hat{A}_{jk}$ to be the $(N-2) \times (N-2)$ matrix with the jth and kth rows and the jth and kth columns removed.

Example 4.48. $\triangleright$ If

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

then

$$\hat{A}_{12} = \begin{bmatrix} \cancel{0} & \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{-a} & \cancel{0} & \cancel{d} & \cancel{e} \\ \cancel{-b} & \cancel{-d} & 0 & f \\ \cancel{-c} & \cancel{-e} & -f & 0 \end{bmatrix}.$$

$\triangleleft$

Proposition 4.49 (Reduction formula for Pfaffians)**.** *Let A be a skew-symmetric $N \times N$ matrix. Then*

$$\mathrm{Pf}(A) = \sum_{k=2}^{n} (-1)^k A_{1k} \mathrm{Pf}(\hat{A}_{1k}).$$

Proof. See [**Pal2**], appendix A.4. $\square$

Remark 4.50. This recursive formula requires a base case. Directly from definition 4.42 we have

$$\mathrm{Pf} \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a.$$

Example 4.51. $\triangleright$ Again consider a general 4×4 skew-symmetric matrix

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

The reduction formula from proposition 4.49 gives

$$a\,\mathrm{Pf} \begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} - b\,\mathrm{Pf} \begin{bmatrix} 0 & e \\ -e & 0 \end{bmatrix} + c\,\mathrm{Pf} \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} = af - be + cd$$

as expected. $\triangleleft$

Proposition 4.52 (Block formula for Pfaffians)**.** *Let A be an $M \times M$ matrix; let B be an $N \times N$ matrix which is also skew-symmetric and invertible. Then*

$$\mathrm{Pf} \begin{bmatrix} B & A \\ -A^\tau & 0 \end{bmatrix} = \mathrm{Pf}(-B)\mathrm{Pf}(-A^\tau B^{-1} A)$$

Proof. TBD. Or cite. □

4.9 Exterior exponentials

Definition 4.53. Let $q \in \Lambda^2(Y)$ where Y is an N-dimensional vector space. Define

$$\exp(q) = \sum_{k=0}^{N/2} \frac{q^k}{k!}$$

where q^k denotes q wedged with itself k times, and $q^0 = 1$. Note that the sum terminates as the top level since the exterior algebra is finite-dimensional.

Definitions 4.54. Let Y be a complex vector space with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$. Let $\sigma \subseteq \{1, \ldots, N\}$, with $1 \leq \sigma_1 < \ldots < \sigma_k \leq N$. Define

$$\mathbf{e}_\sigma = \mathbf{e}_{\sigma_1} \wedge \cdots \wedge \mathbf{e}_{\sigma_k}.$$

Also define P_σ to be the projection from Y onto the span of $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$.

Example 4.55. ▷ For $N = 4$ and $\sigma = \{1, 2\}$, we have

$$\mathbf{e}_\sigma = \mathbf{e}_1 \wedge \mathbf{e}_2 \qquad \text{and} \qquad P_\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

◁

Proposition 4.56. *Let Y be an N-dimensional vector space with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$. Let L be a skew-symmetric $N \times N$ matrix. Then*

$$\exp \frac{1}{2} \sum_{i,j} L_{ij} \mathbf{e}_i \wedge \mathbf{e}_j = \sum_{\sigma \subseteq 2^{\{1,\ldots,N\}}} \mathrm{Pf}(P_\sigma L P_\sigma) e_\sigma.$$

Proof. TBD. □

Example 4.57. ▷ Let $N = 4$ and consider the quadratic form

$$q = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4.$$

One may verify by inspection that this is the bivector associated to the matrix

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned}\exp(q) &= 1 + q + \frac{1}{2}\, q \wedge q \\ &= 1 + (\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4) + \frac{1}{2}\, (\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4) \wedge (\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4) \\ &= 1 + (\mathbf{e}_1 \wedge \mathbf{e}_2) + (\mathbf{e}_3 \wedge \mathbf{e}_4) + (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4).\end{aligned}$$

Recall from proposition 4.46 that the Pfaffian of an odd-dimensional matrix is zero, so we need to consider only σ's having 0, 2, or 4 elements. These are

$$\{\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}.$$

From the matrix Q we see that only the σ's $\{\}$, $\{1, 2\}$, $\{3, 4\}$, and $\{1, 2, 3, 4\}$ contribute non-zero terms. For $\sigma = \{1, 2\}$ and $\{3, 4\}$ we have

$$P_\sigma Q P_\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which has a Pfaffian of 1. The 4×4 matrix Q has Pfaffian $af - be + cd = 1 - 0 + 0 = 1$. Thus, we have

$$\sum_{\sigma \subseteq 2^{\{1,\ldots,N\}}} \mathrm{Pf}(P_\sigma L P_\sigma) e_\sigma \quad = \quad 1 + \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4 + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$$

as expected. $\triangleleft$

This example shows that the Pfaffian serves here as a **coefficient extractor** as discussed in example 4.25.

4.10 Isotropic subspaces and polarizations

This section provides machinery for Fock representations as presented in section 4.11. Again suppose that $(\cdot, \cdot)$ is a non-degenerate bilinear form on a complex vector space Y.

Definition 4.58. A subspace W of Y is **isotropic** if $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u}, \mathbf{v} \in W$.

Remark 4.59. In particular, this means $(\mathbf{u}, \mathbf{u}) = 0$ for all $\mathbf{u} \in W$. Thus an isotropic subspace can have dimension higher than 0 only when the bilinear form $(\cdot, \cdot)$ is not positive-definite.

Example 4.60. $\triangleright$ Let $Y = \mathbb{C}^2$, with bilinear form $(\mathbf{u}, \mathbf{v}) = u_1 v_1 + u_2 v_2$. In an attempt to find an isotropic subspace, let's first find a non-zero vector $\mathbf{w}$ which is orthogonal to itself. The condition $w_1^2 + w_2^2 = 0$ forces $\mathbf{w} = 0$ over $\mathbb{R}$, but over $\mathbb{C}$ we have more freedom. Take $\mathbf{w} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Then $(\mathbf{w}, \mathbf{w}) = 1 - 1 = 0$. Now take W to be the span of $\mathbf{w}$. For any $\mathbf{u} = a\mathbf{w}$ and $\mathbf{v} = b\mathbf{w}$ in W (with $a, b \in \mathbb{C}$) we have

$$(\mathbf{u}, \mathbf{v}) = \left(\begin{pmatrix} a \\ ia \end{pmatrix}, \begin{pmatrix} b \\ ib \end{pmatrix} \right) = ab - ab = 0.$$

Thus W is an isotropic subspace for Y. Note that the non-positive-definiteness of the bilinear form was needed to construct a non-trivial isotropic subspace. $\triangleleft$

Definition 4.61. An **isotropic splitting** or **polarization** of Y is a decomposition of Y into the direct sum of two subspaces $Y = A \oplus B$ where A and B are isotropic subspaces of Y.

Proposition 4.62. *Suppose we have a polarization $Y = A \oplus B$ where the distinguished bilinear form is non-degenerate. Then $A \cong B^*$ (and vice versa) with isomorphism $\mathbf{a} \mapsto (\mathbf{a}, \cdot)$.*

Proof. The map $\mathbf{a} \mapsto (\mathbf{a}, \cdot)$ is linear due to the bilinearity of the form.

For injectivity, let $\mathbf{a}_1, \mathbf{a}_2 \in A$ and suppose $(\mathbf{a}_1, \cdot) = (\mathbf{a}_2, \cdot)$. Then for all $\mathbf{b} \in B$, and recalling the bilinearity of the form, we have

$$\begin{aligned} (\mathbf{a}_1, \mathbf{b}) &= (\mathbf{a}_2, \mathbf{b}) \\ (\mathbf{a}_1, \mathbf{b}) - (\mathbf{a}_2, \mathbf{b}) &= 0 \\ (\mathbf{a}_1 - \mathbf{a}_2, \mathbf{b}) &= 0. \end{aligned}$$

Since the form is non-degenerate and $(\mathbf{a}_1 - \mathbf{a}_2, \mathbf{b}) = 0$ for all $\mathbf{b}$, $\mathbf{a}_1 - \mathbf{a}_2 = 0$ i.e. $\mathbf{a}_1 = \mathbf{a}_2$.

For surjectivity, recall that for finite-dimensional vector spaces we have

$$\dim(A) = \dim(A^*) \qquad \text{and} \qquad \dim(B) = \dim(B^*).$$

The injectivity argument above, along with its obvious analogue for a map from B into A^*, shows that

$$\dim(A) \leq \dim(B^*) \qquad \text{and} \qquad \dim(B) \leq \dim(A^*).$$

From these facts we conclude that

$$\dim(A^*) \leq \dim(B^*) \qquad \text{and} \qquad \dim(B^*) \leq \dim(A^*)$$

which forces

$$\dim(A^*) = \dim(B^*) \qquad \text{and} \qquad \dim(A) = \dim(B).$$

$\square$

Corollary 4.63. *A polarized space is even-dimensional.*

Example 4.64. $\triangleright$ Continuing the previous example, we can seek a complementary isotropic subspace W' in addition to the isotropic subspace W already found. In an attempt to satisfy the condition $Y = W \oplus W'$ — which in particular requires that $W \cap W' = \{0\}$ — we might try looking for the space perpendicular to W using the usual complex inner product. We compute

$$\begin{aligned} \begin{pmatrix} 1 \\ i \end{pmatrix} &\perp \begin{pmatrix} a \\ b \end{pmatrix} \\ \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle &= 0 \\ a - ib &= 0. \end{aligned}$$

Setting $b = -i$ as a parameter we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The subspace W' is seen to be isotropic just as W was. Thus we have the polarization $\mathbb{C}_2 = Y = W \oplus W'$ where

$$W = \left\langle \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \right\rangle \qquad \text{and} \qquad W' = \left\langle \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \right\rangle.$$

$\triangleleft$

Ad-hoc examples notwithstanding, one asks whether one may always obtain an isotropic splitting for a given (even-dimensional by corollary 4.63) vector space Y. We require a lemma before answering this in the affirmative.

Lemma 4.65. *Given a non-degenerate bilinear form on a finite-dimensional complex vector space Y, there exists a basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_N\}$ for Y which is orthonormal with respect to the form.*

Proof. The proof is not meaningfully different from that of the Gram-Schmidt process for non-degenerate sesquilinear forms. See any linear algebra text for details. $\square$

Proposition 4.66. *Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ be a basis for an N-dimensional complex vector space Y, where N is assumed to be even. Further assume that the basis is orthonormal (via lemma 4.65) with respect to the non-degenerate bilinear form. Define*

$$Y_+ = \langle \mathbf{b}_j + i\mathbf{b}_{j+N/2} \rangle \quad \textit{and} \quad Y_- = \langle \mathbf{b}_j - i\mathbf{b}_{j+N/2} \rangle.$$

Then $Y = Y_+ \oplus Y_-$ is a polarization of Y.

Proof. To see that $Y_+ \cap Y_- = \{0\}$, suppose $\mathbf{y} \in Y_+ \cap Y_-$, i.e.

$$\mathbf{y} = \sum_{j=1}^{N/2} c_j (\mathbf{b}_j + i\mathbf{b}_{j+N/2}) = \sum_{j=1}^{N/2} d_j (\mathbf{b}_j - i\mathbf{b}_{j+N/2}).$$

Then

$$\begin{aligned} \sum_{j=1}^{N/2} c_j (\mathbf{b}_j + i\mathbf{b}_{j+N/2}) &= \sum_{j=1}^{N/2} d_j (\mathbf{b}_j - i\mathbf{b}_{j+N/2}) \\ \sum_{j=1}^{N/2} (c_j - d_j)\mathbf{b}_j + i \sum_{j=1}^{N/2} (c_j + d_j)\mathbf{b}_{j+N/2} &= 0. \end{aligned}$$

The linear independence of the $\mathbf{b}_j$'s forces $c_j = d_j$ and $c_j = -d_j$ for all j, hence $c_j = d_j = 0$ and thus $\mathbf{y} = 0$.

To see that $Y_+ + Y_- = Y$, ... let

$$\mathbf{y} = \sum_{j=1}^{N} c_j \mathbf{b}_j \in Y.$$

Then

$$\begin{aligned} \mathbf{y} = \sum_{j=1}^{N} c_j \mathbf{b}_j &= \sum_{j=1}^{N/2} (c_j \mathbf{b}_j + c_{j+N/2} \mathbf{b}_{j+N/2}) \\ &= \sum_{j=1}^{N/2} \left(\frac{c_j - ic_{j+N/2}}{2} (\mathbf{b}_j + i\mathbf{b}_{j+N/2}) + \frac{c_j + ic_{j+N/2}}{2} (\mathbf{b}_j - i\mathbf{b}_{j+N/2}) \right). \end{aligned}$$

To see that Y_+ is an istropic subspace of Y, let

$$\mathbf{y} = \sum_{j=1}^{N/2} c_j (\mathbf{b}_j + i\mathbf{b}_{j+N/2}) \in Y_+.$$

Then

$$
\begin{aligned}
(\mathbf{y},\mathbf{y}) &= \left(\sum_{j=1}^{N/2} c_j(\mathbf{b}_j + i\mathbf{b}_{j+N/2}), \sum_{k=1}^{N/2} c_k(\mathbf{b}_k + i\mathbf{b}_{k+N/2})\right) \\
&= \sum_{j=1}^{N/2}\sum_{k=1}^{N/2} c_j c_k \left(\mathbf{b}_j + i\mathbf{b}_{j+N/2}, \mathbf{b}_k + i\mathbf{b}_{k+N/2}\right) \\
&= \sum_{j=1}^{N/2}\sum_{k=1}^{N/2} c_j c_k \left[(\mathbf{b}_j, \mathbf{b}_k) + i(\mathbf{b}_j, \mathbf{b}_{k+N/2}) + i(\mathbf{b}_{j+N/2}, \mathbf{b}_k) - (\mathbf{b}_{j+N/2}, \mathbf{b}_{k+N/2})\right] \\
&= \sum_{j=1}^{N/2}\sum_{k=1}^{N/2} c_j c_k \left[\delta_{jk} + 0 + 0 - \delta_{jk}\right] \\
&= 0.
\end{aligned}
$$

The proof that Y_- is an istropic subspace of Y is all but identical. □

This orthonormalization process is one technique to construct a polarization of a given vector space. Another technique (using *polarization operators*) is shown in appendix A.2 of [**Pal2**].

4.11 Fock representations

Definition 4.67. Let Y be an even-dimensional complex vector space with a polarization

$$Y = Y_+ \oplus Y_-.$$

In particular, for all $v \in Y$ there are $v_+ \in Y_+$ and $v_- \in Y_-$ such that

$$v = v_+ + v_-.$$

Recall from proposition 4.62 that we may identify Y_- with Y_+^* via

$$v_- \mapsto (\cdot, v_-).$$

Define $F : \mathrm{Cliff}(Y) \to \mathrm{Cliff}(Y)$ by

$$F(v) = c(v_+) + a(v_-).$$

Note that we have $v_+ \in Y_+$, $v_- \in Y_+^*$, and thus (recalling the definitions of the creation and annihilation operators in section 4.6), we have

$$F(v) : \Lambda(Y_+) \to \Lambda(Y_+).$$

Since the creation and annihilation operators are linear transformations on $\Lambda(Y_+)$ (see proposition 4.33), we have

$$F : Y \to \mathrm{Hom}(\Lambda(Y_+)).$$

That is, F is a **representation** of Y on the representation space $\Lambda(Y_+)$. We can extend F to be a representation of $\mathrm{Cliff}(Y)$ on $\Lambda(Y_+)$ if only the Clifford-algebra relations (see definition 4.27) can be satisfied. (We will re-use the letter F.)

Proposition 4.68. *The map $F : Y \to \mathrm{Hom}(\Lambda(Y_+))$ from definition 4.67 satisifies the Clifford-algebra relations from definition 4.27), with composition as multiplication. That is, for all $u, v \in Y$,*

$$F(u)F(v) + F(v)F(u) = (u, v).$$

Proof. From the definition of F, we have

$$\begin{aligned}
F(u)F(v) + F(v)F(u) &= (c(u_+) + a(u_-)) \circ (c(v_+) + a(v_-)) + (c(v_+) + a(v_-)) \circ (c(u_+) + a(u_-)) \\
&= c(u_+) \circ c(v_+) + c(u_+) \circ a(v_-) + a(u_-) \circ c(v_+) + a(u_-) \circ a(v_-) \\
&+ c(v_+) \circ c(u_+) + c(v_+) \circ a(u_-) + a(v_-) \circ c(u_+) + a(v_-) \circ a(u_-).
\end{aligned}$$

Now, using the first two anticommutation relations in proposition 4.36, we have $c(u_+) \circ c(v_+) + c(v_+) \circ c(u_+) = 0$ etc. Thus

$$\begin{aligned}
F(u)F(v) + F(v)F(u) &= c(u_+) \circ a(v_-) + a(u_-) \circ c(v_+) + c(v_+) \circ a(u_-) + a(v_-) \circ c(u_+) \\
&= a(u_-) \circ c(v_+) + c(v_+) \circ a(u_-) + a(v_-) \circ c(u_+) + c(u_+) \circ a(v_-).
\end{aligned}$$

Using the third anticommutation relation in proposition 4.36, we have

$$\begin{aligned}
F(u)F(v) + F(v)F(u) &= u_-(v_+) + v_-(u_+) \\
&= (u_-, v_+) + (v_-, u_+) \\
&= (u_-, v_+) + (u_+, v_-)
\end{aligned}$$

by the symmetry of the bilinear form. This is the same as (u, v). Using the fact that $u = u_+ + u_-$ and $v = v_+ + v_-$, along with the bilinearity of the form, we have

$$(u, v) = (u_+ + u_-, v_+ + v_-) = (u_+, v_+) + (u_+, v_-) + (u_-, v_+) + (u_-, v_-).$$

Since Y is polarized, (u_+, v_+) and (u_-, v_-) are both zero, and so

$$(u, v) = (u_+, v_-) + (u_-, v_+)$$

which is the same as $F(u)F(v) + F(v)F(u)$, which is what was to be shown. □

Definition 4.69. The map $F : \mathrm{Cliff}(Y) \to \Lambda(Y_+)$ as defined in this section is called the **Fock representation** of $\mathrm{Cliff}(Y)$ with respect to the polarization $Y = Y_+ \oplus Y_-$ on the representation space $\Lambda(Y_+)$.

4.12 Iterated commutators

In this section, let A and B be linear transformations from a complex vector space Y to itself.

Definition 4.70. Recall that we define the **commutator** of B with A to be $[A, B] = AB - BA$.

Lemma 4.71. *The commutator is bilinear. That is, for all A, B, C, D, and all scalars λ, μ, we have*

$$[A + B, C + D] = [A, C] + [A, D] + [B, C] + [B, D].$$

and

$$[\lambda A, \mu B] = \lambda\mu[A, B].$$

Proof. For the first claim, we have

$$\begin{aligned}
[A + B, C + D] &= (A + B)(C + D) - (C + D)(A + B) \\
&= AC + AD + BC + BD - CA - CB - DA - DB \\
&= AC - CA + AD - DA + BC - CB + BD - DB \\
&= [A, C] + [A, D] + [B, C] + [B, D].
\end{aligned}$$

For the second claim, we have

$$\begin{aligned}
[\lambda A, \mu B] &= \lambda A \mu B - \mu B \lambda A \\
&= \lambda \mu AB - \lambda \mu BA \\
&= \lambda \mu (AB - BA) \\
&= \lambda \mu [A, B].
\end{aligned}$$

□

Definition 4.72. Define the n-fold **iterated commutator** of B with A, written $\mathrm{ad}^n(A)B$, as

$$\begin{aligned}
\mathrm{ad}^0(A)B &= B \\
\mathrm{ad}^n(A)B &= [A, \mathrm{ad}^{n-1}(A)B] = [A, [A, \ldots, [A, B]]].
\end{aligned}$$

Remark 4.73. In particular, we have the following:

$$\begin{aligned}
\mathrm{ad}^0(A)B &= B \\
\mathrm{ad}^1(A)B &= AB - BA \\
\mathrm{ad}^2(A)B &= [A, [A, B]] = [A, AB - BA] = A(AB - BA) - (AB - BA)A \\
&= A^2B - 2ABA + BA^2 \\
\mathrm{ad}^3(A)B &= A(A^2B - 2ABA + BA^2) - (A^2B - 2ABA + BA^2)A \\
&= A^3B - 2A^2BA + ABA^2 - A^2BA + 2ABA^2 - BA^3 \\
&= A^3B - 3A^2BA + 3ABA^2 - BA^3.
\end{aligned}$$

These computations suggest the following proposition.

Proposition 4.74. *Using the notation from the above definition,*

$$\mathrm{ad}^n(A)B = \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^{n-k} B A^k.$$

Proof. The proof is by induction on n. The base case, $n = 0$, is given by

$$\sum_{k=0}^{0} (-1)^k \binom{n}{k} A^{n-k} B A^k = B = \mathrm{ad}^0(A)B.$$

For the induction step, suppose the result holds for n. Then we have

$$\begin{aligned}
\mathrm{ad}^{n+1}(A)B &= [A, \mathrm{ad}^n(A)B] \\
&= \left[A, \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^{n-k} B A^k \right].
\end{aligned}$$

Using the bilinearity of the commutator (lemma 4.71),

$$\begin{aligned}
\mathrm{ad}^{n+1}(A)B &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[A, A^{n-k} B A^k \right] yy \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(A(A^{n-k} B A^k) - (A^{n-k} B A^k) A \right) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(A^{n+1-k} B A^k - A^{n-k} B A^{k+1} \right).
\end{aligned}$$

Expanding terms to find the pattern:

$$\begin{aligned}
= & \binom{n}{0}A^{n+1}B - \binom{n}{1}A^nBA + \binom{n}{2}A^{n-1}BA^2 - \dots \\
& +(-1)^{n-2}\binom{n}{n-2}A^3BA^{n-2} + (-1)^{n-1}\binom{n}{n-1}A^2BA^{n-1} + (-1)^n\binom{n}{n}ABA^n \\
& -\binom{n}{0}A^nBA + \binom{n}{1}A^{n-1}BA^2 - \binom{n}{2}A^{n-2}BA^3 + \dots \\
& -(-1)^{n-2}\binom{n}{n-2}A^2BA^{n-1} - (-1)^{n-1}\binom{n}{n-1}ABA^n - (-1)^n\binom{n}{n}BA^{n+1}.
\end{aligned}$$

Collecting like terms:

$$\begin{aligned}
\mathrm{ad}^{n+1}(A)B = & \binom{n}{0}A^{n+1}B - \left(\binom{n}{0}+\binom{n}{1}\right)A^nBA + \left(\binom{n}{1}+\binom{n}{2}\right)A^{n-1}BA^2 \\
& -\left(\binom{n}{2}+\binom{n}{3}\right)A^{n-2}BA^3 + \dots \\
& +(-1)^{n-1}\left(\binom{n}{n-2}+\binom{n}{n-1}\right)A^2BA^{n-1} + (-1)^n\left(\binom{n}{n-1}+\binom{n}{n}\right)ABA^n \\
& +(-1)^n\binom{n}{n}BA^{n+1}.
\end{aligned}$$

At this point we need a little combinatorial lemma.

Lemma 4.75. *Adjacent binomial coefficients obey*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Proof. Using the factorial definition of binomial coefficients and acquiring a common denominator, we obtain

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\
&= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} \\
&= \frac{n!((k+1)-(n-k))}{(k+1)!(n-k)!} = \frac{(n+1)!}{(k+1)!((n+1)-(k+1))!} \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

□

Using the lemma, and recalling that

$$\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$$

as well as $-(-1)^k = (-1)^{k+1}$, we have

$$\begin{aligned}
\mathrm{ad}^{n+1}(A)B &= \binom{n+1}{0}A^{n+1}B - \binom{n+1}{1}A^nBA + \binom{n+1}{2}A^{n-1}BA^2 - \binom{n+1}{3}A^{n-2}BA^3 + \dots \\
&\quad +(-1)^{n-1}\binom{n+1}{n-1}A^2BA^{n-1} + (-1)^n\binom{n+1}{n}ABA^n + (-1)^{n+1}\binom{n+1}{n+1}BA^{n+1} \\
&= \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}A^{n+1-k}BA^k
\end{aligned}$$

as desired. □

In the proof of theorem 9.13 in section 9.3 we will use iterated commutators to evaluate an expression of the form $e^{\lambda A}Be^{-\lambda A}$. Writing out the matrix exponentials in Taylor series about $\lambda = 0$ we have:

$$\begin{aligned}
e^{\lambda A} &= I + \lambda A + \lambda^2A^2/2! + \lambda^3A^3/3! + \dots \\
e^{\lambda A B} &= B + \lambda AB + \lambda^2A^2B/2! + \lambda^3A^3B/3! + \dots \\
e^{-\lambda A} &= I - \lambda A + \lambda^2A^2/2! - \lambda^3A^3/3! + \dots .
\end{aligned}$$

FOILing out the product $e^{\lambda A}Be^{-\lambda A}$ in the following tableau (freely re-arranging the terms of the sum) enables us to formulate a conjecture:

	I	$-\lambda A$	$\lambda^2A^2/2!$	$-\lambda^3A^3/3!$	$+\dots$
B	B	$-\lambda BA$	$+\lambda^2BA^2/2!$	$-\lambda^3BA^3/3!$	$\dots$
λAB	$+\lambda AB$	$-\lambda^2ABA$	$+\lambda^3ABA^2/2!$	$-\lambda^4ABA^3/3!$	$\dots$
$\lambda^2A^2B/2!$	$+\lambda^2A^2B/2!$	$-\lambda^3A^2BA/2!$	$+\lambda^4A^2BA^2/2!2!$	$-\lambda^5A^2BA^3/2!3!$	$\dots$
$\lambda^3A^3B/3!$	$+\lambda^3A^3B/3!$	$-\lambda^4A^3BA/3!$	$+\lambda^5A^3BA^2/3!2!$	$-\lambda^6A^3BA^3/3!3!$	$\dots$
$+\dots$	.				

Collecting like powers of λ on anti-diagonals suggests the following proposition.

Proposition 4.76. *Let A and B be linear transformations from a complex vector space Y to itself. Then*

$$e^{\lambda A}Be^{-\lambda A} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathrm{ad}^n(A)B$$

Proof. Writing out the Taylor expansions of $e^{\lambda A}$ and $e^{-\lambda A}$ about $\lambda = 0$ and freely re-arranging terms gives

$$\begin{aligned}
(e^{\lambda A}B)e^{-\lambda A} &= \left(\sum_{i=0}^{\infty}\lambda^iA^iB/i!\right)\left(\sum_{j=0}^{\infty}(-1)^j\lambda^jA^j/j!\right) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\lambda^{i+j}A^iBA^j/i!j! \\
&= \sum_{n=0}^{\infty}\sum_{i+j=n}(-1)^j\lambda^nA^iBA^j/i!j!.
\end{aligned}$$

Since the inner sum is over $i + j = n$, we may put $k = j$ and $i = n - k$ to obtain

$$\begin{aligned}
(e^{\lambda A}B)e^{-\lambda A} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\lambda^n\frac{A^{n-k}BA^k}{(n-k)!k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\lambda^n\frac{n!A^{n-k}BA^k}{n!(n-k)!k!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\frac{\lambda^n}{n!}\binom{n}{k}A^{n-k}BA^k \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}A^{n-k}BA^k.
\end{aligned}$$

By proposition 4.74, the inner sum is precisely $\mathrm{ad}^n(A)B$ and so

$$(e^{\lambda A}B)e^{-\lambda A} = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\mathrm{ad}^n(A)B$$

as desired. $\square$

5 Continuous and discrete operators

5.1 Symmetric and skew-symmetric operators

Let X be a vector space, real or complex, and not necessarily finite-dimensional. In particular, we might consider X to be $\mathbb{R}^n$, $\mathbb{C}^n$, or a Hilbert function space such as (real- or complex-valued) $L^2(-\infty, +\infty)$ or $L^2[0,1]$ or $L^2[a,b]$.

Definition 5.1. A **linear operator** is simply a linear transformation from X to X.

Example 5.2. $\triangleright$ For finite-dimensional vector spaces, we usually think of operators as matrices. $\triangleleft$

Example 5.3. $\triangleright$ For function spaces, canonical examples include:

- $f(x) \mapsto df/dx$.
- $f(x) \mapsto d^2 f/dx^2$.
- $f(x) \mapsto x\, f(x)$.
- $f(x) \mapsto \int K(x,y) f(y) dy$.

$\triangleleft$

Definition 5.4. Now suppose X has a distinguished bilinear form $(\cdot,\cdot)$, and let L be a linear operator on X. Then L is said to be **symmetric** if

$$(Lf, g) = (f, Lg)$$

for all $f, g \in X$. Likewise, L is said to be **skew-symmetric** if

$$(Lf, g) = -(f, Lg)$$

for all $f, g \in X$.

Remark 5.5. On finite-dimensional vector spaces, there is no difference between symmetric and self-adjoint operators. See [**Lax**] for more information.

Example 5.6. $\triangleright$ Matrices are symmetric or skew-symmetric as operators if they are symmetric or skew-symmetric, respectively, in coordinates: $A_{ij} = A_{ji}$ or $A_{ij} = -A_{ji}$, respectively. $\triangleleft$

Definition 5.7. For function spaces, define the symmetric bilinear form

$$(f, g) = \int f(x) g(x) dx.$$

Example 5.8. $\triangleright$ Let $X = L^2[a,b]$ and let

$$W = \{f \in X : f(a) = f(b) = 0\}.$$

The operator $L = d/dx$ is skew-symmetric on W, which is to say

$$(Lf, g) = -(f, Lg)$$

i.e.

$$\int_a^b \frac{df}{dx}\, g(x)\, dx + \int_a^b f(x)\, \frac{dg}{dx}\, dx = 0.$$

To see this, integrate by parts:

$$\int_a^b \frac{df}{dx}\, g(x)\, dx + \int_a^b f(x)\, \frac{dg}{dx}\, dx = f(b)g(b) - f(a)g(a) = 0$$

due to the boundary conditions on W. $\triangleleft$

Example 5.9. $\triangleright$ Let X and W be as in the previous example. Let $M = d^2/dx^2$. Then M is symmetric on W, i.e. $(Mf, g) - (f, Mg) = 0$. To see this, integrate by parts on each term and apply the boundary conditions:

$$\begin{aligned}
(Mf,g) - (f,Mg) &= \int_a^b f''(x)g(x)dx - \int_a^b f(x)g''(x)dx \\
&= \left([f'(x)g(x)]_a^b - \int_a^b f'(x)g'(x)dx \right) - \left([f(x)g'(x)]_a^b - \int_a^b f'(x)g'(x)dx \right) \\
&= [f'(x)g(x)]_a^b - [f(x)g'(x)]_a^b = 0.
\end{aligned}$$

$\triangleleft$

Remark 5.10. In the previous two examples, L and M are not skew-symmetric and symmetric, respectively, on all of X. One may think of the symmetric bilinear form

$$(f,g)_L = (Lf,g) + (f,Lg),$$

which is identically zero on W, as measuring the lack of skew-symmetry of L on X. Likewise, the skew-symmetric bilinear form

$$(f,g)_M = (Mf,g) - (f,Mg),$$

measures the lack of symmetry of M on X.

Proposition 5.11. *Let X, L, and M be as above. L and M are indeed bilinear forms.*

Proof. Use the linearity of derivative and integral:

$$\begin{aligned}
(af_1 + f_2, bg_1 + g_2)_L &= (af_1' + f_2', bg_1 + g_2) + (af_1 + f_2, bg_1' + g_2') \\
&= ab(f_1', g_1) + ab(f_1', g_2) + ab(f_2', g_1) + ab(f_2', g_2) \\
&+ ab(f_1, g_1') + ab(f_1, g_2') + ab(f_2, g_1') + ab(f_2, g_2') \\
&= ab(f_1, g_1)_L + ab(f_1, g_2)_L + ab(f_2, g_1)_L + ab(f_2, g_2)_L
\end{aligned}$$

as well as

$$\begin{aligned}
(af_1 + f_2, bg_1 + g_2)_M &= (af_1'' + f_2'', bg_1 + g_2) - (af_1 + f_2, bg_1'' + g_2'') \\
&= ab(f_1'', g_1) + ab(f_1'', g_2) + ab(f_2'', g_1) + ab(f_2'', g_2) \\
&- ab(f_1, g_1'') - ab(f_1, g_2'') - ab(f_2, g_1'') - ab(f_2, g_2') \\
&= ab(f_1, g_1)_M + ab(f_1, g_2)_M + ab(f_2, g_1)_M + ab(f_2, g_2)_M.
\end{aligned}$$

For symmetry and skew-symmetry:

$$\begin{aligned}
(f,g)_L &= (f',g) + (f,g') = (g',f) + (g,f') = (g,f)_L, \\
(f,g)_M &= (f'',g) - (f,g'') = -(g'',f) + (g,f'') = -(g,f)_M.
\end{aligned}$$

$\square$

5.2 Discrete operators

Recall that the derivatives of a real-valued function are often, in practice, approximated numerically by finite differences:

$$
\begin{aligned}
f'(x) &\approx \frac{f(x+h)-f(x)}{h} \quad \text{(Right-difference rule)} \\
f'(x) &\approx \frac{f(x)-f(x-h)}{h} \quad \text{(Left-difference rule)} \\
f'(x) &\approx \frac{f(x+h)-f(x-h)}{2h} \quad \text{(Centered-difference rule).}
\end{aligned}
$$

For this paper, the centered-difference rule will be used. Likewise, for the second derivative, we naturally use:

$$
f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.
$$

The same approximation technique applies for directional derivatives, in the direction of the vector $\mathbf{h}$, of functions from $\mathbb{R}^n$ to $\mathbb{R}$:

$$
\begin{aligned}
D_{\mathbf{h}} f(\mathbf{x}) &\approx \frac{f(\mathbf{x}+\mathbf{h})-f(\mathbf{x}-\mathbf{h})}{2\|\mathbf{h}\|} \\
D^2_{\mathbf{h}} f(\mathbf{x}) &\approx \frac{f(\mathbf{x}+\mathbf{h})-2f(\mathbf{x})+f(\mathbf{x}-\mathbf{h})}{\|\mathbf{h}\|^2}.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\frac{df}{dx_i} &\approx \frac{f(\mathbf{x}+h\mathbf{e}_i)-f(\mathbf{x}-h\mathbf{e}_i)}{2h} \\
\frac{d^2 f}{dx_i^2} &\approx \frac{f(\mathbf{x}+h\mathbf{e}_i)-2f(\mathbf{x})+f(\mathbf{x}-h\mathbf{e}_i)}{h^2}.
\end{aligned}
$$

Now consider $\mathbf{f}:\mathbb{R}\to\mathbb{R}^n$. One approximates as before:

$$
\begin{aligned}
\mathbf{f}'(x) &\approx \frac{\mathbf{f}(x+h)-\mathbf{f}(x-h)}{2h} \\
\mathbf{f}''(x) &\approx \frac{\mathbf{f}(x+h)-2\mathbf{f}(x)+\mathbf{f}(x-h)}{h^2}.
\end{aligned}
$$

For numerical work, one often replaces the continuous domain with a **mesh**, i.e. an evenly spaced sequence of x_k's. For the case $\mathbf{f}:\mathbb{R}\to\mathbb{R}^n$, the estimated first and second derivatives become:

$$
\begin{aligned}
\mathbf{f}'(x_k) &\approx \frac{1}{2h}\left(\mathbf{f}(x_{k+1})-\mathbf{f}(x_{k-1})\right) \\
\mathbf{f}''(x_k) &\approx \frac{1}{h^2}\left(\mathbf{f}(x_{k+1})-2\mathbf{f}(x_k)+\mathbf{f}(x_{k-1})\right).
\end{aligned}
$$

We may formalize these operations in terms of vector spaces and operators by thinking of the input sequence $\{\mathbf{f}(x_k)\}$ as being in, say, $X=\ell^\infty(\mathbb{R}^n)$. Then we have output sequences $\{\mathbf{y}(x_k)\approx\mathbf{f}'(x_k)\}$ and $\{\mathbf{z}(x_k)\approx$

$\mathbf{f}''(x_k)\}$ also in X. We obtain linear operators L and M on X via

$$
\begin{aligned}
\mathbf{y}(x) &= L\mathbf{f}(x) \\
\begin{bmatrix} \vdots \\ \mathbf{y}(x_{k-1}) \\ \mathbf{y}(x_k) \\ \mathbf{y}(x_{k+1}) \\ \vdots \end{bmatrix} &= \frac{1}{2h} \begin{bmatrix} \ddots & \ddots & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{f}(x_{k-1}) \\ \mathbf{f}(x_k) \\ \mathbf{f}(x_{k+1}) \\ \vdots \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{z}(x) &= M\mathbf{f}(x) \\
\begin{bmatrix} \vdots \\ \mathbf{z}(x_{k-1}) \\ \mathbf{z}(x_k) \\ \mathbf{z}(x_{k+1}) \\ \vdots \end{bmatrix} &= \frac{1}{h^2} \begin{bmatrix} \ddots & \ddots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbf{f}(x_{k-1}) \\ \mathbf{f}(x_k) \\ \mathbf{f}(x_{k+1}) \\ \vdots \end{bmatrix}
\end{aligned}
$$

The central point of these examples is that $L = d/dx$ and $M = d^2/dx^2$ were skew-symmetric and symmetric (respectively) as continuous operators, while their discretized versions are skew-symmetric and symmetric (respectively) as matrices.

Discrete domains and discrete difference operators are discussed in more detail in section 6, with this section having provided a little motivation.

6 Discrete vector bundles

6.1 Infinite bundles

Definition 6.1. Let V be a finite-dimensional complex vector space. Consider the set of functions

$$\tilde{V}(\mathbb{Z}) = \{f : \mathbb{Z} \to V\}.$$

Define an inner product $\langle \cdot, \cdot \rangle$ on $\tilde{V}(\mathbb{Z})$ using the standard inner product on V:

$$\langle f, g \rangle = \sum_{k \in \mathbb{Z}} \langle f(k), g(k) \rangle.$$

This induces a norm on $\tilde{V}(\mathbb{Z})$ via

$$\|f\|^2 = \langle f, f \rangle.$$

Define $V(\mathbb{Z})$ to be the **square-summable functions**:

$$V(\mathbb{Z}) = \{f : \mathbb{Z} \to V : \langle f, f \rangle < \infty.\}$$

That is, $V(\mathbb{Z}) = \ell^2(V)$. We call this a **discrete vector bundle**; we say that V is the **model** for the bundle.

Remark 6.2. We assume we have an orthogonal space and with a basis of $\mathbf{e}_i$'s which are orthgonal with respect to the form.

Definition 6.3. We define $\psi_j(k)$ to be $\mathbf{e}_j$ with support only on the kth fiber.

Definition 6.4. Define a **symmetric bilinear form** on $V(\mathbb{Z})$:

$$(f, g) = \sum_{k \in \mathbb{Z}} (f(k), g(k))$$

where on the right-hand side we use the distinguished bilinear form on V.

Definition 6.5. We define

$$\omega_k = \psi_1(k) \wedge \cdots \wedge \psi_N(k).$$

and

$$\omega_\Omega = \prod_{k \in \Omega} \omega_k.$$

6.2 Truncated bundles

Definition 6.6. For a finite subset U of $\mathbb{Z}$, define $V(U)$ to be the subspace of $V(\mathbb{Z})$ consisting of functions with support in U.

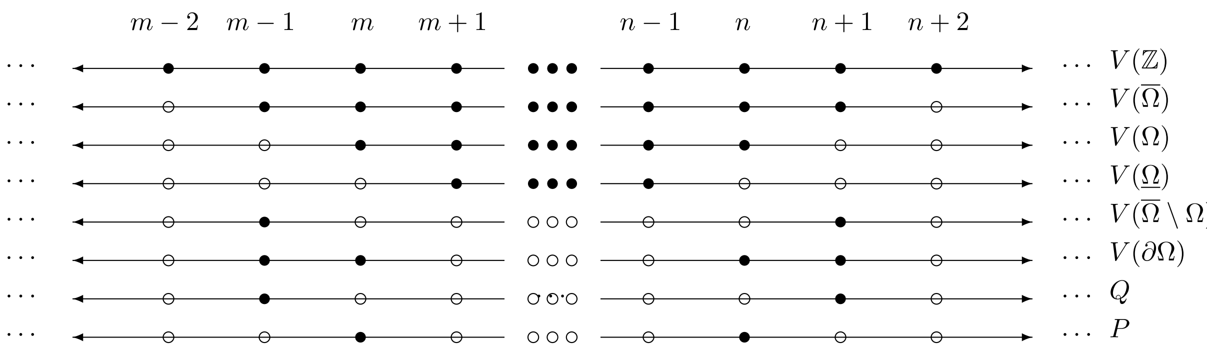
Remark 6.7. Due to the finiteness of U, *all* functions with support on U are square-summable.

A few particular finite subsets of $\mathbb{Z}$ are of interest for this paper. Let $m < n \in \mathbb{Z}$. We define

- $\Omega = \{m, m+1, \ldots, n-1, n\}$.
- $\overline{\Omega} = \{m-1, m, m+1, \ldots, n-1, n, n+1\}$.

- $\underline{\Omega} = \{m+1, \ldots, n-1\}$.
- $\overline{\Omega} \setminus \Omega = \{m-1, n+1\}$.
- $\Omega \setminus \underline{\Omega} = \{m+1, n-1\}$.
- $\partial\Omega = \{m-1, m, n, n+1\}$.
- $P = V(\Omega \setminus \underline{\Omega})$.
- $Q = V(\overline{\Omega} \setminus \Omega)$.
- $V(\partial\Omega) = P \oplus Q$.

These may be visualized as follows:



7 Discrete difference operators

7.1 Difference operators

Accompany the discrete vector bundle by **connection matrices** $A(k+\frac{1}{2})$ and $B(k)$. The former are indexed by half-integers, which is intended to be reminiscent of connecting bonds between nodes.

Definition 7.1. Let $L : V(\mathbb{Z}) \to V(\mathbb{Z})$ be given by

$$(Lf)(k) = A(k+\tfrac{1}{2})f(k+1) + B(k)f(k) - A^\tau(k-\tfrac{1}{2})f(k-1).$$

We take the transpose A^τ with respect to the distinguished bilinear form. (As discussed in remark 4.16, this is trivial to compute (using the standard matrix transpose) as long as we use a basis which is orthonormal with respect to the form.) Also, we assume $B(k) = -B^\tau(k)$. As noted in [**Pal1**], this L is intended to be the most general range-one finite-difference operator.

Applying the definition to a few fibers yields:

$$\begin{array}{lcllllllll}
 & \vdots & & & & & & & & \\
(Lf)(k-1) & = & A_{k-\frac{1}{2}} & f(k) & + & B(k-1) & f(k-1) & - & A^\tau(k-\frac{3}{2}) & f(k-2) \\
(Lf)(k) & = & A_{k+\frac{1}{2}} & f(k+1) & + & B(k) & f(k) & - & A^\tau(k-\frac{1}{2}) & f(k-1) \\
(Lf)(k+1) & = & A_{k+\frac{3}{2}} & f(k+2) & + & B(k+1) & f(k+1) & - & A^\tau(k+\frac{1}{2}) & f(k) \\
 & \vdots & . & & & & & & &
\end{array}$$

Writing an ordered basis for $V(\mathbb{Z})$ as

$$\{\ldots, \psi_1(k-1), \ldots, \psi_N(k-1), \quad \psi_1(k), \ldots, \psi_N(k), \quad \psi_1(k+1), \ldots, \psi_N(k+1), \ldots\},$$

we may write L as an infinite-dimensional tri-block-diagonal matrix:

$$\begin{bmatrix} \vdots \\ (Lf)(k-1) \\ (Lf)(k) \\ (Lf)(k+1) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & B_{k-1} & A_{k-\frac{1}{2}} & 0 & \\ \ddots & -A^\tau{}_{k-\frac{1}{2}} & B_k & A_{k+\frac{1}{2}} & \ddots \\ & 0 & -A^\tau{}_{k+\frac{1}{2}} & B_{k+1} & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ f(k-1) \\ f(k) \\ f(k+1) \\ \vdots \end{bmatrix}.$$

Proposition 7.2. *L is formally skew-symmetric with respect to the distinguished bilinear form.*

Proof. This is visually obvious. $\square$

7.2 Truncated difference operators

Definition 7.3. For $U \subseteq \mathbb{Z}$, define P_U to be the **projection** of $V(\mathbb{Z})$ onto $V(U)$. (Note: this extends definitions 4.54.)

We have the following matrices:

$$P_\Omega = \left[\begin{array}{cc|cccc|cc} \cdots & 0 & 1 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & 0 & \cdots \end{array}\right] \quad \text{and} \quad P_{\overline{\Omega}\setminus\Omega} = \left[\begin{array}{c|c|c} 1 & 0 \quad\quad & \\ \hline & \quad\quad 0 & 1 \end{array}\right] \begin{array}{l} \text{(Output index } m-1) \\ \text{(Output index } n+1) \end{array}$$

Note that LP_U restricts input, making a vertical strip; $P_U L$ restricts output, making a horizontal strip. We have

$$P_\Omega f = \begin{bmatrix} f(m) \\ f(m+1) \\ \vdots \\ f(n-1) \\ f(n) \end{bmatrix} \qquad \text{and} \qquad P_{\overline{\Omega}\setminus\Omega} f = \begin{bmatrix} f(m-1) \\ f(n+1) \end{bmatrix}.$$

Likewise,

$$P_\Omega L f = \begin{bmatrix} (Lf)(m) \\ (Lf)(m+1) \\ \vdots \\ (Lf)(n-1) \\ (Lf)(n) \end{bmatrix} \qquad \text{and} \qquad P_{\overline{\Omega}\setminus\Omega} L f = \begin{bmatrix} (Lf)(m-1) \\ (Lf)(n+1) \end{bmatrix}.$$

Thus we write L as an infinite matrix:

$$L = \left[\begin{array}{ll|l|llll|l|ll} & & (m-1) & (m) & \cdots & \cdots & (n) & (n+1) & \text{Input} & \\ & \ddots & \ddots & & & & & & \text{index} & \\ \hline (m-1) & {-A^{\tau}}_{m-\frac{3}{2}} & B_{m-1} & A_{m-\frac{1}{2}} & & & & & & \\ \hline (m) & & {-A^{\tau}}_{m-\frac{1}{2}} & B_m & A_{m+\frac{1}{2}} & & & & & \\ (m+1) & & & {-A^{\tau}}_{m+\frac{1}{2}} & B_{m+1} & A_{m+\frac{3}{2}} & & & & \\ \vdots & & & & \ddots & \ddots & \ddots & & & \\ (n-1) & & & & {-A^{\tau}}_{n-\frac{3}{2}} & B_{n-1} & A_{n-\frac{1}{2}} & & & \\ (n) & & & & & {-A^{\tau}}_{n-\frac{1}{2}} & B_n & A_{n+\frac{1}{2}} & & \\ \hline (n+1) & & & & & & {-A^{\tau}}_{n+\frac{1}{2}} & B_{n+1} & A_{n+\frac{3}{2}} & \\ \hline \text{Output} & \text{index} & & & & & & \ddots & \ddots & \ddots \end{array}\right]$$

As well:

$$P_\Omega L = \left[\begin{array}{l|llll|l} 0 & 0 & & & & \\ \hline {-A^{\tau}}_{m-\frac{1}{2}} & B_m & A_{m+\frac{1}{2}} & & & \\ & {-A^{\tau}}_{m+\frac{1}{2}} & B_{m+1} & A_{m+\frac{3}{2}} & & \\ & & \ddots & \ddots & \ddots & \\ & & {-A^{\tau}}_{n-\frac{3}{2}} & B_{n-1} & A_{n-\frac{1}{2}} & \\ & & & {-A^{\tau}}_{n-\frac{1}{2}} & B_n & A_{n+\frac{1}{2}} \\ \hline & & & & 0 & 0 \end{array}\right]$$

and

$$
LP_\Omega = \left[\begin{array}{c|cccc|c}
0 & A_{m-\frac{1}{2}} & & & & \\
\hline
0 & B_m & A_{m+\frac{1}{2}} & & & \\
 & -A^\tau{}_{m+\frac{1}{2}} & B_{m+1} & A_{m+\frac{3}{2}} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & -A^\tau{}_{n-\frac{3}{2}} & B_{n-1} & A_{n-\frac{1}{2}} & \\
 & & & -A^\tau{}_{n-\frac{1}{2}} & B_n & 0 \\
\hline
 & & & & -A^\tau{}_{n+\frac{1}{2}} & 0
\end{array}\right].
$$

Foreshadowing our attempt to do integrals which examine boundary values, we can decompose the **truncated operator** $L_{\overline{\Omega}}$ into the sum of four matrices:

$$
P_\Omega LP_\Omega = \left[\begin{array}{c|cccc|c}
0 & 0 & & & & \\
\hline
0 & B_m & A_{m+\frac{1}{2}} & & & \\
 & -A^\tau{}_{m+\frac{1}{2}} & B_{m+1} & A_{m+\frac{3}{2}} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & -A^\tau{}_{n-\frac{3}{2}} & B_{n-1} & A_{n-\frac{1}{2}} & \\
 & & & -A^\tau{}_{n-\frac{1}{2}} & B_n & 0 \\
\hline
 & & & & 0 & 0
\end{array}\right]
$$

$$
P_\Omega LP_{\overline{\Omega}\backslash\Omega} = \left[\begin{array}{c|cccc|c}
0 & 0 & & & & \\
\hline
-A^\tau{}_{m-\frac{1}{2}} & 0 & 0 & & & \\
 & 0 & 0 & 0 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & 0 & 0 & 0 & \\
 & & & 0 & 0 & A_{n+\frac{1}{2}} \\
\hline
 & & & & 0 & 0
\end{array}\right]
$$

$$
P_{\overline{\Omega}\backslash\Omega} LP_\Omega = \left[\begin{array}{c|cccc|c}
0 & A_{m-\frac{1}{2}} & & & & \\
\hline
0 & 0 & 0 & & & \\
 & 0 & 0 & 0 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & 0 & 0 & 0 & \\
 & & & 0 & 0 & 0 \\
\hline
 & & & & -A^\tau{}_{n+\frac{1}{2}} & 0
\end{array}\right]
$$

$$
P_{\overline{\Omega}\backslash\Omega} LP_{\overline{\Omega}\backslash\Omega} = \left[\begin{array}{c|cccc|c}
B_{m-1} & 0 & & & & \\
\hline
0 & 0 & 0 & & & \\
 & 0 & 0 & 0 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & 0 & 0 & 0 & \\
 & & & 0 & 0 & 0 \\
\hline
 & & & & 0 & B_{n+1}
\end{array}\right]
$$

Definition 7.4. We can now define the matrix of the truncated skew-symmetric operator $L_{\overline{\Omega}}$ relative to the splitting $V(\overline{\Omega}) = V(\Omega) \oplus V(\overline{\Omega} \setminus \Omega)$:

$$
\begin{aligned}
L_{\overline{\Omega}} &= \begin{bmatrix} P_\Omega L P_\Omega & P_\Omega L P_{\overline{\Omega}\setminus\Omega} \\ P_{\overline{\Omega}\setminus\Omega} L P_\Omega & 0 \end{bmatrix} \\
&= \left[\begin{array}{c|cccc|c}
0 & A_{m-\frac{1}{2}} & & & & \\
\hline
-A^\tau{}_{m-\frac{1}{2}} & B_m & A_{m+\frac{1}{2}} & & & \\
 & -A^\tau{}_{m+\frac{1}{2}} & B_{m+1} & A_{m+\frac{3}{2}} & & \\
 & & \ddots & \ddots & \ddots & \\
 & & -A^\tau{}_{n-\frac{3}{2}} & B_{n-1} & A_{n-\frac{1}{2}} & \\
 & & & -A^\tau{}_{n-\frac{1}{2}} & B_n & A_{n+\frac{1}{2}} \\
\hline
 & & & & -A^\tau{}_{n+\frac{1}{2}} & 0
\end{array}\right]
\end{aligned}
$$

Note that we omit $P_{\overline{\Omega}\setminus\Omega} L P_{\overline{\Omega}\setminus\Omega}$. This is intentional, and is done solely to be able to make use, in section 8.2, of the Pfaffian block formula which was presented in section 4.52.

Remark 7.5. It will be assumed below, but is not required by anything written so far, that $P_\Omega L P_\Omega$ is invertible, but that $P_\Omega L$ has a nullspace.

7.3 The action functional

See [**Fra**] for more information about action functionals.

Definition 7.6. Given a skew-symmetric operator L as above, we define

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}} [L\psi_j(k)] \wedge \psi_j(k).$$

Remark 7.7. This sum will not converge unless truncated. This will be done below.

Proposition 7.8. *$\mathcal{L}$ is formally independent of the choice of orthonormal basis for V.*

Proof. TBD. ☐

Definition 7.9. Given $\mathcal{L}$ as above, we define the **truncated action**

$$\mathcal{L}_{\overline{\Omega}} = \frac{1}{2} \sum_{k \in \overline{\Omega}} \sum_{j=1}^{N} [L_{\overline{\Omega}}\psi_j(k)] \wedge \psi_j(k).$$

Note that $\mathcal{L}$ is the **associated bivector** for the matrix L, in the sense of definition 4.39.

Definition 7.10. Recalling section 4.9,

$$\exp(\mathcal{L}) = 1 + \mathcal{L} + \frac{\mathcal{L} \wedge \mathcal{L}}{2} + \ldots + \frac{\mathcal{L}^{N/2}}{(N/2)!}$$

As presented, computing this would be a mess of wedge products. Pfaffians will be used below to faciliate the computation.

8 The discrete Berezin integral

8.1 The discrete Berezin integral and the vacuum vector

The discrete Berezin integral on $\Lambda(V(\overline{\Omega}))$ is defined by picking out the coefficient of a top-level form in $\Lambda(V(\Omega))$. This is not an integral per se, but rather a functional. Nonetheless, this boundary behavior is vaguely reminiscent of the fundamental theorem of calculus.

Definition 8.1. Recall definition 6.5 of ω_Ω. Then we define

$$B_\Omega(\alpha\omega_\Omega + \text{lower-order terms}) = \alpha.$$

and in particular for the action functional of section 7.3 we write

$$v_\Omega = \mathcal{B}_\Omega(\exp \mathcal{L}_{\overline{\Omega}}).$$

Example 8.2. $\triangleright$ Let $N = 2$ and

$$y = \psi_1(m-1) \wedge \Big[\psi_1(m) \wedge \psi_2(m)\Big] \wedge \cdots \wedge \Big[\psi_1(n) \wedge \psi_2(n)\Big] \wedge \psi_1(n+1) \wedge \psi_2(n+1).$$

Then, since

$$\Big[\psi_1(m) \wedge \psi_2(m)\Big] \wedge \cdots \wedge \Big[\psi_1(n) \wedge \psi_2(n)\Big] = \omega_\Omega,$$

we have

$$B_\Omega(y) = \psi_1(m-1) \wedge \psi_1(n+1) \wedge \psi_2(n+1).$$

$\triangleleft$

It is clear that this discrete integral obeys the rules described in section 2.5.

8.2 Computation of the vacuum vector

Pfaffians were developed in section 4.8 in order to facilitate the following computation of the vacuum vector defined in section 8.1. First, some notation is presented.

In section 6.2 I presented a diagram of $V(\Omega)$ where each fiber was represented by a single bead. Here I zoom in and think of each fiber as an N-dimensional vector space. I denote each one-dimensional subspace of each fiber with a dot, gathering the one-dimensional subspaces of each fiber into a box, and then gathering all of $V(\Omega)$ into a bigger box. This leaves $V(\overline{\Omega} \setminus \Omega) = Q$ on the left and right sides:

Given this picture, the Berezin integrand is a form on $V(\overline{\Omega})$ which is fully supported on all of $V(\Omega)$. The result of the integral will be a wedge product taken from among the $2N$ basis vectors for Q. Since we are integrating the exponential of a *quadratic* form (the action functional $\mathcal{L}$ as presented in section 7.3), the wedge products in the result of the integral will consist only of even-degree terms. Using the work-avoidance principle which I exhorted in example 4.25, I would like to find the coefficients of each wedge term without needing to write out high-dimensional arrays.

Here we can use proposition proposition 4.56, as illustrated by example 4.57. Since our integrand has full support on $V(\Omega)$, the formula of proposition 4.56 becomes

$$v_\Omega = \mathcal{B}_\Omega(\exp \mathcal{L}_{\overline{\Omega}}) = \sum_\sigma \operatorname{Pf} \begin{bmatrix} P_\Omega L P_\Omega & P_\Omega L P_\sigma \\ P_\sigma L P_\Omega & 0 \end{bmatrix} e_\sigma$$

where we let the σ's range only over the $2N$ boundary values.

Now we apply the the Pfaffian block formula in proposition 4.52: if B is skew-symmetric and invertible, then

$$\operatorname{Pf} \begin{bmatrix} B & A \\ -A^\tau & 0 \end{bmatrix} = \operatorname{Pf}(-B)\operatorname{Pf}(-A^\tau B^{-1} A).$$

We defined $L_{\overline{\Omega}}$, in definition 7.4, to have zeroes at the upper left and lower right corners. We also assumed in remark 7.5 that the matrix $P_\Omega L P_\Omega$ is invertible. Both decisions were made precisely so that proposition 4.52 applies and we have, with A being $P_\Omega L P_\sigma$ and B being $P_\Omega L P_\omega$:

$$\begin{aligned} \operatorname{Pf} \begin{bmatrix} P_\Omega L P_\Omega & P_\Omega L P_\sigma \\ P_\sigma L P_\Omega & 0 \end{bmatrix} &= \operatorname{Pf}(-P_\Omega L P_\Omega)\operatorname{Pf}(-(P_\Omega L P_\sigma)^\tau (P_\Omega L P_\Omega)^{-1} P_\Omega L P_\sigma) \\ &= \operatorname{Pf}(-P_\Omega L P_\Omega)\operatorname{Pf}(P_\sigma L P_\Omega (P_\Omega L P_\Omega)^{-1} P_\Omega L P_\sigma). \end{aligned}$$

Then, since clearly $P_\sigma P_{\overline{\Omega}\setminus\Omega} = P_\sigma$:

$$v_\Omega = \mathcal{B}_\Omega(\exp \mathcal{L}_{\overline{\Omega}}) = \sum_\sigma \operatorname{Pf}(P_\sigma L P_\Omega (P_\Omega L P_\Omega)^{-1} P_\Omega L P_\sigma) e_\sigma.$$

Definition 8.3. Define

$$X = P_{\overline{\Omega}\setminus\Omega} L P_\Omega (P_\Omega L P_\Omega)^{-1} P_\Omega L P_{\overline{\Omega}\setminus\Omega}.$$

Proposition 8.4. *X is a skew-symmetric map on Q.*

Proof. TBD. ☐

Given this definition of X, we now have

$$\begin{aligned} v_\Omega &= \operatorname{Pf}(-P_\Omega L P_\Omega) \sum_\sigma \operatorname{Pf}(P_\sigma X P_\sigma) e_\sigma \\ &= \operatorname{Pf}(-P_\Omega L P_\Omega)\ \exp\left(\frac{1}{2}\sum [X\mathbf{e}_j] \wedge \mathbf{e}_j\right). \end{aligned}$$

9 The Fock characterization

9.1 The symmetric bilinear form

We will construct a Clifford algebra (as presented in section 4.5) using the symmetric bilinear form presented here.

Definition 9.1. Recall the definitions of $P_\Omega L$ and LP_Ω from section 7.2. Then

$$S = P_\Omega L - LP_\Omega = \left[\begin{array}{c|cccc|c} 0 & -A_{m-\frac{1}{2}} & & & & \\ \hline -A^{\tau}{}_{m-\frac{1}{2}} & 0 & 0 & & & \\ & 0 & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 0 & \\ & & & 0 & 0 & A_{n+\frac{1}{2}} \\ \hline & & & & A^{\tau}_{n+\frac{1}{2}} & 0 \end{array}\right].$$

This form is designed to measure the lack of skew-symmetry of $P_\Omega L$, as motivated in remark 5.10.

On $V(\partial\Omega)$, S is:

$$\begin{bmatrix} (Sf)(m-1) & \in & Q \\ (Sf)(m) & \in & P \\ (Sf)(n) & \in & P \\ (Sf)(n+1) & \in & Q \end{bmatrix} = \begin{bmatrix} 0 & -A_{m-\frac{1}{2}} & 0 & 0 \\ -A^{\tau}{}_{m-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{n+\frac{1}{2}} \\ 0 & 0 & A^{\tau}_{n+\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} f(m-1) & \in & Q \\ f(m) & \in & P \\ f(n) & \in & P \\ f(n+1) & \in & Q \end{bmatrix}$$

Proposition 9.2. *S as given in definition 9.1 is a symmetric linear transformation on $V(\partial\Omega)$.*

Proof. This is obvious from the picture. □

Proposition 9.3. *$S : P \to Q$ and $S : Q \to P$.*

Proof. This is obvious from the picture. □

Definition 9.4. Define a form on $P \oplus Q$ via

$$(f, g)_S = (Sf, g)$$

using the bilinear form $(\cdot, \cdot)$ on V defined in definition 6.4.

Proposition 9.5. *The form in definition 9.4 is symmetric and bilinear.*

Proof. Let $f, g \in P \oplus Q$. For symmetry, observe that

$$\begin{aligned} (f,g)_S &= (Sf, g) && \text{(definition of } S\text{)} \\ &= (g, Sf) && \text{(symmetry of } (\cdot,\cdot)\text{)} \\ &= (Sg, f) && \text{(symmetry of } S\text{)} \\ &= (g, f)_S && \text{(definition of } S\text{).} \end{aligned}$$

For bilinearity, we have

$$\begin{aligned}
(f_1 + f_2, g)_S &= (S(f_1 + f_2), g) \qquad \text{(definition of } S\text{)} \\
&= (S(f_1) + S(f_2), g) \qquad \text{(linearity of } S\text{)} \\
&= S(f_1), g + S(f_2), g \qquad \text{(bilinearity of } (\cdot,\cdot)\text{)} \\
&= (f_1, g)_S + (f_2, g)_S \qquad \text{(definition of } S\text{).}
\end{aligned}$$

Right-linearity and left/right scalar multiplicativity are all but identical and are omitted. ☐

Remark 9.6. Since S is symmetric, i.e. $S = S^\tau$, we have

$$(Sf, g) = (f, S^\tau g) = (f, Sg).$$

for all $f, g \in P \oplus Q$.

Proposition 9.7. *S invertible implies $(\cdot,\cdot)_S$ is non-degenerate.*

Proof. TBD. ☐

Remark 9.8. We assume (see [**Pal1**]) that S is invertible, so that $(\cdot,\cdot)_S$ is non-degenerate.

9.2 The Fock representation

Recall from section 4.11 that we may form a Fock representation of $V(\partial\Omega) = P \oplus Q$ if the vector space is even-dimensional with a non-degenerate symmetric bilinear form and a polarization. Since P and Q are of the same dimension, the first criterion is satisfied; the second criterion is satisfied by $(\cdot,\cdot)_S$ (proposition 9.5).

Proposition 9.9. *The decomposition $V(\partial\Omega) = P \oplus Q$ is a polarization with respect to $(\cdot,\cdot)_S$.*

Proof. $V(\partial\Omega) = P \oplus Q$ is already a direct-sum decomposition. It remains to show that $(\cdot,\cdot)_S$ is zero on P and Q. Let $f_P \in P \oplus Q$ be supported only in P. Then

$$(f_P, f_P)_S = (Sf_P, f_P).$$

Recall from proposition 9.3 that $Sf_P \in Q$. From definition 6.4, the form evaluated on an element supported solely in Q and an element supported solely in P yields zero. The proof for the case when f_Q is supported only in Q is analogous. ☐

Definition 9.10. Let $f \in V(\partial\Omega) = P \oplus Q$. Then there exist $f_P \in P$ and $f_Q \in Q$ such that $f = f_P + f_Q$. View f_Q as an element of Q^* via $f_Q \mapsto (\cdot, f_Q)_S$. Define $\Psi : P \oplus Q \to \mathrm{Hom}(\Lambda(Q))$ by

$$\Psi(f) = \Psi(f_P + f_Q) = c(Sf_P) + a(f_Q).$$

Remark 9.11. Since $S : P \to Q$ and $S : Q \to P$ (proposition 9.3), we have

$$c(Sf_P) \in \mathrm{Hom}(\Lambda(Q))$$

and

$$a(f_Q) \in \mathrm{Hom}(\Lambda(Q)).$$

In section 4.11 we identified $v_- \in Y_-$ with an element of Y_+^*, where here P and Q play the roles of Y_- and Y_+, respectively. Here, though, we identify f_Q with an element of Q^* rather than P^*. This is because $(f_P, g)_S = (Sf_P, g)$ sends f_P from P to Q before the bilinear form is applied.

Proposition 9.12. *The map Ψ of definition 9.10 satisfies the Clifford-algebra relations of definition 4.27), i.e. for $f, g \in P \oplus Q$,*

$$\Psi(f)\Psi(g) + \Psi(g)\Psi(f) = (f, g)_S.$$

Proof. Let $f = f_P + f_Q$ and $g = g_P + g_Q$. Then

$$\begin{aligned}
\Psi(f)\Psi(g) + \Psi(g)\Psi(f) &= (c(Sf_P) + a(f_Q)) \circ (c(Sg_P) + a(g_Q)) \\
&+ (c(Sg_P) + a(g_Q)) \circ (c(Sf_P) + a(f_Q)) \\
&= c(Sf_P) \circ c(Sg_P) + c(Sf_P) \circ a(g_Q) + a(f_Q) \circ c(Sg_P) + a(f_Q) \circ a(g_Q) \\
&+ c(Sg_P) \circ c(Sf_P) + c(Sg_P) \circ a(f_Q) + a(g_Q) \circ c(Sf_P) + a(g_Q) \circ a(f_Q).
\end{aligned}$$

Using the first two anticommutation relations from proposition 4.36, we have

$$\begin{aligned}
\Psi(f)\Psi(g) + \Psi(g)\Psi(f) &= c(Sf_P) \circ a(g_Q) + a(f_Q) \circ c(Sg_P) \\
&+ c(Sg_P) \circ a(f_Q) + a(g_Q) \circ c(Sf_P) \\
&= a(f_Q) \circ c(Sg_P) + c(Sg_P) \circ a(f_Q) \\
&+ a(g_Q) \circ c(Sf_P) + c(Sf_P) \circ a(g_Q).
\end{aligned}$$

Using the third anticommutation relation from proposition 4.36, we have

$$\Psi(f)\Psi(g) + \Psi(g)\Psi(f) = (f_Q, Sg_P) + (g_Q, Sf_P).$$

On the other hand,

$$\begin{aligned}
(f, g)_S &= (f_P + f_Q, g_P + g_Q)_S \qquad \text{(given)} \\
&= (f_P, g_P)_S + (f_P, g_Q)_S + (f_Q, g_P)_S + (f_Q, g_Q)_S \qquad \text{(bilinearity of } S\text{)} \\
&= (Sf_P, g_P) + (Sf_P, g_Q) + (Sf_Q, g_P) + (Sf_Q, g_Q) \qquad \text{(definition of } S\text{)} \\
&= (Sf_P, g_Q) + (Sf_Q, g_P) \qquad \text{(polarization of } P \oplus Q\text{)} \\
&= (f_P, Sg_Q) + (f_Q, Sg_P) \qquad \text{(symmetry of } S\text{)} \\
&= \Psi(f)\Psi(g) + \Psi(g)\Psi(f)
\end{aligned}$$

which is what was to be shown. □

Thus, as in section 4.11,

$$\Psi : \mathrm{Cliff}(P \oplus Q) \to \Lambda(Q)$$

is a **Fock representation** associated with the symmetric form $(\cdot, \cdot)_S$.

9.3 Vacuum characterization

Theorem 9.13. *The vacuum vector*

$$v_\Omega = \mathcal{B}_\Omega(e^{\mathcal{L}_{\overline{\Omega}}})$$

is projectively characterized (i.e. up to a constant multiplier) as the unique vector in $\Lambda(Q)$ that is annihilated by all elements $\Psi(f)$ where f is the boundary value in $P \oplus Q$ of a function $F \in V(\overline{\Omega})$ in the nullspace of $P_\Omega L$. That is,

$$\Psi(f)v_\Omega = 0$$

whenever $f = F|_{\partial\Omega}$ and

$$P_\Omega LF = 0.$$

Proof. (Note: The presentation here will complete the sketch given in [**Pal1**].)

Step 1. Recall from section 8.2 that the vacuum vector we are trying to characterize is

$$v_\Omega = \exp\left(\frac{1}{2}\sum_{j=1}^{2N} X\mathbf{e}_j \wedge \mathbf{e}_j\right).$$

Using the bra-ket notation from section 4.7, and in particular using remark 4.38, this may be written as

$$v_\Omega = \exp\left(\frac{1}{2}\sum_{j=1}^{2N} c(X\mathbf{e}_j)c(\mathbf{e}_j)|\,0\,\rangle\right) = \exp\left(\frac{1}{2}\sum_{j=1}^{2N} c(X\mathbf{e}_j)c(\mathbf{e}_j)\right)|\,0\,\rangle$$

where the exterior arithmetic is done in $\Lambda(Q)$.

Step 2. Now suppose $f \in Q$. Use [do that!] proposition 4.76 to show that

$$\exp\left(\frac{1}{2}\sum_{j=1}^{2N} c(X\mathbf{e}_j)c(\mathbf{e}_j)\right) a(f) \exp\left(-\frac{1}{2}\sum_{j=1}^{2N} c(X\mathbf{e}_j)c(\mathbf{e}_j)\right) = c(Xf) + a(f).$$

Step 3. From this it follows [show that!] that

$$(c(Xf) + a(f))v_\Omega = \exp\left(\frac{1}{2}\sum_{j=1}^{2N} c(X\mathbf{e}_j)c(\mathbf{e}_j)\right) a(f)|\,0\,\rangle = 0 \tag{9.1}$$

for $f \in Q$, since $a(f)|\,0\,\rangle = 0$.

Step 4. Now recall the representation of the Clifford algebra for $(V_{\partial\Omega}, S)$ is given by

$$\Psi(f) = c(Sf_P) + a(f_Q).$$

Suppose that $f \in V_{\partial\Omega}$ is the boundary value of a function $F \in V(\overline{\Omega})$ (i.e. $f = F|_{\partial\Omega}$) with

$$P_\Omega L f = 0. \tag{9.2}$$

Write $N_{\partial\Omega}$ for the subspace of $V_{\partial\Omega}$ which consists of boundary values of functions $F \in V(\overline{\Omega})$ which satisfy equation 9.2. Now suppose that F satisfies 9.2 and write

$$f = F|_{\partial\Omega} = f_P + f_Q$$

and

$$F = F_\Omega + F_Q.$$

The equation which F must satisfy becomes

$$P_\Omega L P_\Omega F_\Omega + P_\Omega L f_Q = 0,$$

from which it follows that

$$F_\Omega = -(P_\Omega L P_\Omega)^{-1} P_\Omega L f_Q. \tag{9.3}$$

Step 5. Recall from remark 7.5 that we suppose $P_\Omega L P_\Omega$ is invertible. Now apply $P_{\overline{\Omega}\setminus\Omega} S$ to both sides of this last equality and recall the definitions of S and X to find

$$Sf_P = Xf_Q.$$

Thus we see that for $f \in N_{\partial\Omega}$ we have

$$\Psi(f) = c(Sf_P) + a(f_Q) = c(Xf_Q) + a(f_Q).$$

From equation 9.1 we see that

$$\Psi(f)v_\Omega = 0 \tag{9.4}$$

for $f \in N_{\partial\Omega}$.

Step 6. It remains to check that $v_{\partial\Omega}$ is projectively characterized by this condition. This turns out to be equivalent to showing that $N_{\partial\Omega}$ is a maximal isotropic subspace of $(V_{\partial\Omega}, S)$. To see this, suppose that $f, g \in N_{\partial\Omega}$ are boundary values of F, G satisfying equation 9.2. Then

$$S(f, g) = (P_\Omega LF, G) + (F, P_\Omega LG) = 0.$$

This shows that $N_{\partial\Omega}$ is isotropic. Because S interchanges Q and P, and because the subspaces are orthogonal to one another with respect to the symmetric form $(\cdot,\cdot)$, it follows that Q and P are complementary istropic subspaces for the symmetric form $(\cdot,\cdot)_S$. Since S is supposed non-degenerate, this means that both Q and P are maximal isotropic subspaces. However, equation 9.3 shows that $N_{\partial\Omega}$ is a graph over Q. Thus it has the same dimension as Q and so must be a maximal isotropic subspace.

Step 7. To finish the proof that equation 9.4 projectively characterizes v_Ω, first show that the vector

$$|\,0\,\rangle = 1 \oplus 0 \oplus \cdot \oplus 0 \in \Lambda(V_+)$$

is projectively characterized by the condition

$$\Psi(f)|\,0\,\rangle = 0$$

for $f \in V_-$, in the Fock representation of the Clifford relations Ψ associated with the polarization $V_+ \oplus V_-$. Then use the equivalence of representations. $\square$

10 Example of vacuum vector

To help fix ideas, we may explicitly compute some of the quantities described in this paper. Take a simple case of $N = 2$ and constant coefficients:

$$A = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}, \quad m = 1, n = 4.$$

Then recall from section 7.2 that we have

$$L_{\overline{\Omega}} = \left[\begin{array}{cc|cc|cc|cc|cc|cc}
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-a & 0 & 0 & b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & -b & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -a & 0 & 0 & b & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a & -b & 0 & 0 & -a & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & -a & 0 & 0 & b & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & -b & 0 & 0 & -a & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & b & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a & -b & 0 & 0 & -a \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0
\end{array}\right]$$

and

$$P_\Omega L P_\Omega = \left[\begin{array}{cc|cc|cc|cc}
0 & b & a & 0 & 0 & 0 & 0 & 0 \\
-b & 0 & 0 & -a & 0 & 0 & 0 & 0 \\
\hline
-a & 0 & 0 & b & a & 0 & 0 & 0 \\
0 & a & -b & 0 & 0 & -a & 0 & 0 \\
\hline
0 & 0 & -a & 0 & 0 & b & a & 0 \\
0 & 0 & 0 & a & -b & 0 & 0 & -a \\
\hline
0 & 0 & 0 & 0 & -a & 0 & 0 & b \\
0 & 0 & 0 & 0 & 0 & a & -b & 0
\end{array}\right].$$

One computes

$$\mathrm{Pf}(-P_\Omega L P_\Omega) = a^4 + 3a^2 b^2 + b^4.$$

One may compute $(P_\Omega L P_\Omega)^{-1}$ using a tool such as Mathematica, but it is too messy to fit on this page. This illustrates the need for further cleverness — we don't need all of this matrix, but rather only its boundary components in X.

Recall that

$$P_{\overline{\Omega}\setminus\Omega} L P_\Omega = \left[\begin{array}{cc|cc|cc|cc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{array}\right]$$

and

$$P_\Omega L P_{\overline{\Omega}\setminus\Omega} = \left[\begin{array}{cc|cc}
-a & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 0 & a & 0 \\
0 & 0 & 0 & -a
\end{array}\right].$$

Then, using definition 8.3 of X and the omitted messy $(P_\Omega L P_\Omega)^{-1}$, we obtain

$$X=\left[\begin{array}{cc|cc} 0 & -\frac{a^2\left(b^3+2a^2b\right)}{a^4+3b^2a^2+b^4} & -\frac{a^5}{a^4+3b^2a^2+b^4} & 0 \\ \frac{a^2\left(b^3+2a^2b\right)}{a^4+3b^2a^2+b^4} & 0 & 0 & \frac{a^5}{a^4+3b^2a^2+b^4} \\ \hline \frac{a^5}{a^4+3b^2a^2+b^4} & 0 & 0 & -\frac{a^2\left(b^3+2a^2b\right)}{a^4+3b^2a^2+b^4} \\ 0 & -\frac{a^5}{a^4+3b^2a^2+b^4} & \frac{a^2\left(b^3+2a^2b\right)}{a^4+3b^2a^2+b^4} & 0 \end{array}\right].$$

Since $N=2$, σ's include the $1,2$ slots of node $m=1$ and the $1,2$ slots of node $n=4$. Here are all 16 σ's, represented schematically with indicator functions:

$\mathbf{e}_{00,00}$	$\mathbf{e}_{00,01}$	$\mathbf{e}_{00,10}$	$\mathbf{e}_{00,11}$
$\mathbf{e}_{01,00}$	$\mathbf{e}_{01,01}$	$\mathbf{e}_{01,10}$	$\mathbf{e}_{01,11}$
$\mathbf{e}_{10,00}$	$\mathbf{e}_{10,01}$	$\mathbf{e}_{10,10}$	$\mathbf{e}_{10,11}$
$\mathbf{e}_{11,00}$	$\mathbf{e}_{11,01}$	$\mathbf{e}_{11,10}$	$\mathbf{e}_{11,11}$

Only even σ's contribute to the Pfaffian. Here are all the even σ's and their corresponding wedges:

$\mathbf{e}_{00,00}=1$	$\mathbf{e}_{00,11}=\psi_1(5)\wedge\psi_2(5)$
$\mathbf{e}_{01,01}=\psi_2(0)\wedge\psi_2(5)$	$\mathbf{e}_{01,10}=\psi_2(0)\wedge\psi_1(5)$
$\mathbf{e}_{10,01}=\psi_1(0)\wedge\psi_2(5)$	$\mathbf{e}_{10,10}=\psi_1(0)\wedge\psi_1(5)$
$\mathbf{e}_{11,00}=\psi_1(0)\wedge\psi_2(0)$	$\mathbf{e}_{11,11}=\psi_1(0)\wedge\psi_2(0)\wedge\psi_1(5)\wedge\psi_2(5)$

One computes

$$\begin{aligned} P_{11,11}XP_{11,11} &= X \\ \mathrm{Pf}(-P_\Omega L P_\Omega) &= a^4+3b^2a^2+b^4 \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{11,00}XP_{11,00}) &= -a^2\left(b^3+2a^2b\right) \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{10,10}XP_{10,10}) &= -a^5 \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{10,01}XP_{10,01}) &= 0 \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{01,10}XP_{01,10}) &= 0 \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{01,01}XP_{01,01}) &= a^5 \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{00,11}XP_{00,11}) &= -a^2\left(b^3+2a^2b\right) \\ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(X) &= a^4\left(a^2+b^2\right). \end{aligned}$$

Using the formula in section 8.2, we have

$$
\begin{aligned}
v_\Omega &= \mathrm{Pf}(-P_\Omega L P_\Omega) \sum_\sigma \mathrm{Pf}(P_\sigma X P_\sigma) e_\sigma \\
&= \mathrm{Pf}(-P_\Omega L P_\Omega)\mathbf{e}_{00,00} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{11,00} X P_{11,00})\mathbf{e}_{11,00} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{10,10} X P_{10,10})\mathbf{e}_{10,10} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{10,01} X P_{10,01})\mathbf{e}_{10,01} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{01,10} X P_{01,10})\mathbf{e}_{01,10} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{01,01} X P_{01,01})\mathbf{e}_{01,01} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(P_{00,11} X P_{00,11})\mathbf{e}_{00,11} \\
&+ \mathrm{Pf}(-P_\Omega L P_\Omega)\mathrm{Pf}(X) \\
&= (a^4 + 3b^2a^2 + b^4) \\
&+ a^2\left(b^3 + 2a^2 b\right)\psi_1(0)\wedge\psi_2(0) \\
&- a^5\psi_1(0)\wedge\psi_1(5) \\
&+ (a^5)\psi_2(0)\wedge\psi_2(5) \\
&- a^2\left(b^3 + 2a^2 b\right)\psi_1(5)\wedge\psi_2(5) \\
&+ (a^4\left(a^2 + b^2\right)\psi_1(0)\wedge\psi_2(0)\wedge\psi_1(5)\wedge\psi_2(5).
\end{aligned}
$$

* * *

This should be equal, up to a constant multiplier, to the vector described in theorem 9.13. To begin to compute this, we first need to find the nullspace of $P_\Omega L$. Recall from the theory of finite difference equations with constant coefficients (such as the Fibonacci sequence) that we guess a solution of the form

$$f(k) = c^k u.$$

We write down

$$
\begin{aligned}
-A^\tau f(k-1) + Bf(k) + Af(k+1) &= 0 \\
-A^\tau c^{k-1}u + Bc^k u + Ac^{k+1}u &= 0 \\
\left(-A^\tau + Bc + Ac^2\right)c^{k-1}u &= 0.
\end{aligned}
$$

Ignore trivial solutions:

$$
\begin{aligned}
\left(-A^\tau + Bc + Ac^2\right)u &= 0 \\
\det\left(-A^\tau + Bc + Ac^2\right) &= 0 \\
-A^\tau + Bc + Ac^2 &= \begin{bmatrix} ac^2 - a & bc \\ -bc & a - ac^2 \end{bmatrix} \\
\det\left(-A^\tau + Bc + Ac^2\right) &= -a^2c^4 + 2a^2c^2 + b^2c^2 - a^2 = 0 \\
a^2c^4 - 2a^2c^2 - b^2c^2 + a^2 &= 0.
\end{aligned}
$$

Solve for c in terms of a and b, using the quadratic formula:

$$\begin{aligned} a^2c^4 - (2a^2+b^2)c^2 + a^2 &= 0 \\ c^2 &= \frac{1}{2}a^2\left(2a^2+b^2 \pm \sqrt{b^4+4a^2b^2}\right) \end{aligned}$$

We have four possibilities for c:

$$\begin{aligned} c_1 &= \frac{\sqrt{\frac{2a^2+b^2-\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ c_2 &= \frac{\sqrt{\frac{2a^2+b^2+\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ c_3 &= -\frac{\sqrt{\frac{2a^2+b^2-\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ c_4 &= -\frac{\sqrt{\frac{2a^2+b^2+\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}}. \end{aligned}$$

To compute the eigenvector u in each case, we examine the matrices $-A^\tau + Bc + Ac^2$:

$$M_1 = \begin{bmatrix} \frac{b^2-\sqrt{b^4+4a^2b^2}}{2a} & \frac{b\sqrt{\frac{2a^2+b^2-\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ -\frac{b\sqrt{\frac{2a^2+b^2-\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} & \frac{\sqrt{b^4+4a^2b^2}-b^2}{2a} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \frac{b^2+\sqrt{b^4+4a^2b^2}}{2a} & \frac{b\sqrt{\frac{2a^2+b^2+\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ -\frac{b\sqrt{\frac{2a^2+b^2+\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} & -\frac{b^2+\sqrt{b^4+4a^2b^2}}{2a} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} \frac{b^2-\sqrt{b^4+4a^2b^2}}{2a} & -\frac{b\sqrt{\frac{2a^2+b^2-\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ \frac{b\sqrt{\frac{2a^2+b^2-\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} & \frac{\sqrt{b^4+4a^2b^2}-b^2}{2a} \end{bmatrix}$$

$$M_4 = \begin{bmatrix} \frac{b^2+\sqrt{b^4+4a^2b^2}}{2a} & -\frac{b\sqrt{\frac{2a^2+b^2+\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} \\ \frac{b\sqrt{\frac{2a^2+b^2+\sqrt{b^4+4a^2b^2}}{a^2}}}{\sqrt{2}} & -\frac{b^2+\sqrt{b^4+4a^2b^2}}{2a} \end{bmatrix}$$

We may then use row reduction to find the eigenvectors.

11 Further directions

The work presented here is a starting point for much further investigation. Possibilities include:

- Compare and contrast symmetric (bosonic) integrals with the skew-symmetric (fermionic) integrals presented here.
- Consider the join of multiple intervals Ω_1 and Ω_2. This may entail examination of the space of all skew-symmetric boundary conditions for the intersection node, using line bundles.
- Restore the continuum limit for the truncated, discretized integral, obtaining an integral on an interval; extend that integral to the real line. This may involve zeta-function regularization for determinants.
- Extend the one-dimensional discussion to two or three dimensions.

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