An argument-settling (?) approach to the Monty Hall problem

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The Monty Hall problem has been discussed to death — see its Wikipedia article for background. Why should I write anything about it?

My first year of graduate school, I saw a few fellow students getting into a heated argument about the Monty Hall problem (MHP). Such behavior, I thought, was unbefitting mathematicians — after all, shouldn't there be a way to settle the issue quickly and irrevocably, where everyone could see the solution and agree with it? I was disinterested in arguing.

A few years later, I saw the problem come up again in a book I was reading, where the author had a brief but unsatisfying proof, and I finally decided to spell it out for myself. There are short, elegant arguments — but these are precisely the kinds of things people argue about. Sometimes, clarity lies in verbosity. For me, at least, the following is as clear as I can make it.

1 The setup

The 1970s game show *Let's Make a Deal* included a final round with the following setup:

- There are three doors. Hidden behind one of them is a car; hidden behind each of the other two is a goat. The game-show host, Monty Hall, knows which is where; the contestant does not.
- (1) The contestant picks one of the three doors. It remains closed.
- (2) Monty opens one of the two other doors — one which contains a goat. Now the contestant sees two closed doors.
- (3) The contestant will choose one of the two remaining doors, staying with their original choice from step (1), or switching to the other door. The door they choose will be opened, and they will win what's behind — the car or the goat.

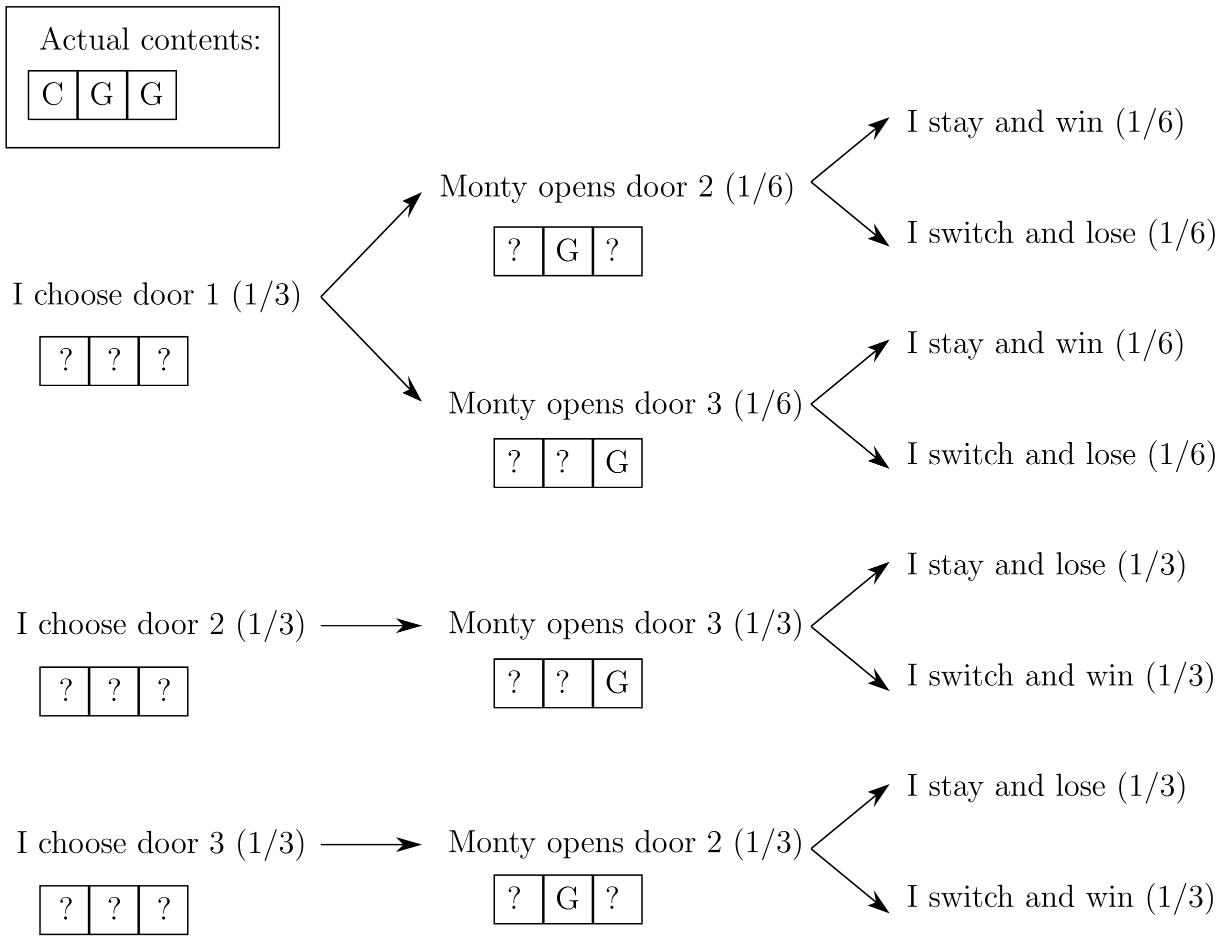

Figure 1: Decision tree for MHP. Decision trees for the cases where the car is behind door 2 or 3 are similar.

Question: Should the contestant stay or switch? Clearly, for the initial choice in step (1), the contestant has a one in three chance of selecting the door with a car behind it. In the long controversy of MHP (again, see the Wikipedia article for details), some people have claimed that Monty's action in step (2) doesn't change the information available to the contestant, so the player's chances remain one in three after a switch.

2 The decision tree

A decision tree, listing in full detail all outcomes which can happen, makes this all transparent. (See figure 1.)

Suppose door 1 has the car and the other two have goats. For shorthand I write [C][G][G] for this configuration. (If door 2 or 3 has the car, then the answer works out the same, as you can check if you like.) To simplify the pronouns, I'll act as though I'm the contestant.

- Monty knows the car is behind door 1 (these are the "Actual contents" in the box in the figure), but I don't. He sees [C][G][G] but I see [?][?][?].
- For step (1), I know nothing, so I choose door 1, 2, or 3 with equal probability 1/3. (More precisely, I should say that no matter which door I choose, there will be a car behind it with probability 1/3.)
- For step (2):
  - If I choose the door with the car, Monty chooses either one of the other two doors, 2 or 3, with equal probability 1/2, since both have goats behind them. I see either [?][G][?] or [?][?][G].
  - If I choose door 2, which Monty knows has a goat behind it, Monty has *no choice* but to open door 3 — that's where the other goat is. I see [?][?][G]. This is the central fact, as will be explained below.
  - Likewise, if I choose door 3, which Monty knows has a goat behind it, Monty must open door 2; I see [?][G][?].
- For step (3): I either stay or switch; the possibilities are listed in the last textual column of the figure.

To figure up my odds if I stay or switch, I can look at that last column. If I stay with my choice of door from step (1), I win in two of four cases and lose in two of four cases. If I switch, I also win in two of four cases and lose in two of four caes. It looks like switching doesn't help.

The crux of the problem — what people end up arguing about so passionately, whether they realize it or not — lies in neglecting the last numerical column in the figure. Namely, not all eight possibilities occur with equal probability; not all carry equal weight. Adding up the probabilities, we see the following:

- 1/3 of the time I open door 1 and 1/2 of those times Monty opens door 2, i.e. this occurs 1/6 of the time.
- 1/3 of the time I open door 1 and 1/2 of those times Monty opens door 3, i.e. this occurs 1/6 of the time.
- 1/3 of the time I open door 2 and Monty must open door 3, i.e. this occurs 1/3 of the time.

- 1/3 of the time I open door 3 and Monty must open door 2, i.e. this occurs 1/3 of the time.

In conclusion:

- If I stay, I win $1/6 + 1/6 = 1/3$ of the time and lose $1/3 + 1/3 = 2/3$ of the time.
- If I switch, I win $1/3 + 1/3 = 2/3$ of the time and lose $1/6 + 1/6 = 1/3$ of the time.

Switching in step (3) improves my chance of winning. And that (I hope) is that!