Lattice quadrupling for percolation in quantum networks

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Outline

Acknowledgements

This works extends *Entanglement Distribution in Pure-State Quantum Networks*, Perseguers, Cirac, Acín, Lewenstein, and Wehr, `arXiv:0708.1025v2`.

An excellent introduction to quantum information is *Quantum Computation and Quantum Information* by Nielsen and Chuang. I also recommend Sakurai's *Modern Quantum Mechanics*, as well as Landau and Binder's *A Guide to Monte Carlo Simulations in Statistical Physics*.

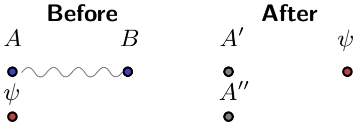
I would like to thank Janek Wehr, Tom Kennedy, and John LaPeyre for multiple insights.

Review of quantum teleportation and entanglement swapping

Quantum teleportation: perfect case

Quantum computation involves manipulation of *qubits*: $\psi = c\,|0\rangle + d\,|1\rangle$ with $|c|^2 + |d|^2 = 1$. Quantum devices require quantum wires: devices to move qubits from point A to point B.

Alice, in possession of qubit ψ at point A, can't measure her qubit; this would collapse (modify) its state. Using local operations and classical communication (LOCC), though, Alice *can* communicate her qubit to Bob.



Ingredients: An *entangled pair* (Bell state) of qubits A and B, e.g. $\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$, a classical wire, and the message qubit ψ.

Quantum teleportation: perfect case

Gates are linear transformations on $\mathbb{C}^2$, $\mathbb{C}^2 \otimes \mathbb{C}^2$, etc. Useful ones here are

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \text{(Hadamard)}, \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{(Kronecker format)}.$$

Quantum teleportation: perfect case

Start with

$$|\psi\rangle \otimes |\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big[c\,|0\rangle + d\,|1\rangle\Big]\Big[\,|00\rangle + \,|11\rangle\Big]$$
$$= \frac{1}{\sqrt{2}}\Big[c\,|0\rangle(\,|00\rangle + \,|11\rangle) + d\,|1\rangle(\,|00\rangle + \,|11\rangle)\Big].$$

Apply CNOT at the first two slots (a local operation for Alice):

$$\frac{1}{\sqrt{2}}\Big[c\,|0\rangle(\,|00\rangle + \,|11\rangle) + d\,|1\rangle(\,|10\rangle + \,|01\rangle)\Big].$$

Apply a Hadamard matrix at the first slot (also a local operation for Alice):

$$\frac{1}{2}\Big[(c\,|0\rangle + c\,|1\rangle)(\,|00\rangle + \,|11\rangle) + (d\,|1\rangle - d\,|0\rangle)(\,|10\rangle + \,|01\rangle)\Big]$$
$$= \frac{1}{2}\Big[(c\,|000\rangle + c\,|011\rangle + c\,|100\rangle + c\,|111\rangle) + (d\,|011\rangle + d\,|000\rangle + d\,|110\rangle + d\,|101\rangle)\Big]$$
$$= \frac{1}{2}\Big[\,|00\rangle(c\,|0\rangle + d\,|1\rangle) + \,|01\rangle(c\,|1\rangle + d\,|0\rangle) + \,|10\rangle(c\,|0\rangle - d\,|1\rangle) + \,|11\rangle(c\,|1\rangle - d\,|0\rangle)\Big].$$

Quantum teleportation: perfect case

Now Alice measures the first two qubits in the basis $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and communicates the result classically to Bob.

- If she obtains $|00\rangle$ then Bob has $c\,|0\rangle + d\,|1\rangle$. Bob applies I and recovers ψ.
- If she obtains $|01\rangle$ then Bob has $c\,|1\rangle + d\,|0\rangle$. Bob applies X and recovers ψ.
- If she obtains $|10\rangle$ then Bob has $c\,|0\rangle - d\,|1\rangle$. Bob applies Z and recovers ψ.
- If she obtains $|11\rangle$ then Bob has $c\,|1\rangle - d\,|0\rangle$. Bob applies $Z \circ X$ and recovers ψ.

Those operations are all local for Bob.

Quantum teleportation: imperfect case

This can be done even with a non-maximally entangled pair of qubits, i.e. $a\,|00\rangle + b\,|11\rangle$ with $|a|^2 + |b|^2 = 1$. But now the message qubit ψ is successfully moved from point A to point B only with singlet conversion probability (SCP) which depends on a and b.

First one converts the pair $a\,|00\rangle + b\,|11\rangle$ into the perfect singlet $\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle$. This succeeds with probability p which is $2(1 - |a|^2)$ if $|a| \leq |b|$, else $2(1 - |b|^2)$.

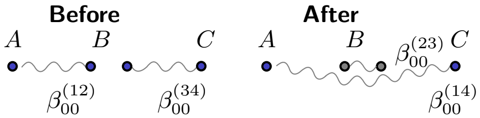
Then, one does quantum teleportation as in the perfect case.

Entanglement swapping: perfect case

The next step toward constructing a quantum network is to chain a pair of links. There are two options.

(1) Simply teleport ψ from A to B, then from B to C.

(2) *Entanglement swapping* changes A-B and B-C links into a B-B link (which is discarded) and an A-C link. Using quantum teleportation, a message qubit ψ may then be moved from point A to point C. Here we discuss only step 1, since step 2 is just as before. Thus, ψ doesn't appear in the figures here.



Which approach is better? That is the key point under discussion today.

How entanglement swapping works: perfect case

Start with $\beta_{00}^{(12)} = \frac{1}{\sqrt{2}}\Big[\,|00\rangle^{(12)} + |11\rangle^{(12)}\Big]$ and $\beta_{00}^{(34)} = \frac{1}{\sqrt{2}}\Big[\,|00\rangle^{(34)} + |11\rangle^{(34)}\Big]$. These maximally entangled pairs are the A-B and B-C links. Then

$$\begin{aligned}
\beta_{00}^{(12)} \otimes \beta_{00}^{(34)} &= \frac{1}{\sqrt{2}}\Big[\,|00\rangle^{(12)} + |11\rangle^{(12)}\Big] \otimes \frac{1}{\sqrt{2}}\Big[\,|00\rangle^{(12)} + |11\rangle^{(12)}\Big] \\
&= \frac{1}{2}\Big[\,|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle\Big] \quad (*).
\end{aligned}$$

The key idea is that Bob, sitting between Alice and Charlie, measures the two qubits in his possession onto the Bell basis which is

$$\begin{aligned}
\beta_{00}^{(ij)} &= \frac{1}{\sqrt{2}}\Big[\,|00\rangle^{(ij)} + |11\rangle^{(ij)}\Big], & \beta_{01}^{(ij)} &= \frac{1}{\sqrt{2}}\Big[\,|01\rangle^{(ij)} + |10\rangle^{(ij)}\Big], \\
\beta_{10}^{(ij)} &= \frac{1}{\sqrt{2}}\Big[\,|00\rangle^{(ij)} - |11\rangle^{(ij)}\Big], & \beta_{11}^{(ij)} &= \frac{1}{\sqrt{2}}\Big[\,|01\rangle^{(ij)} - |10\rangle^{(ij)}\Big].
\end{aligned}$$

To apply that to (*), it's helpful to invert these four equations to find the standard basis in terms of the Bell basis.

How entanglement swapping works: perfect case

We obtain

$$|00\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left[\beta_{00}^{(ij)} + \beta_{10}^{(ij)}\right], \qquad |01\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left[\beta_{01}^{(ij)} + \beta_{11}^{(ij)}\right],$$

$$|10\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left[\beta_{01}^{(ij)} - \beta_{11}^{(ij)}\right], \qquad |11\rangle^{(ij)} = \frac{1}{\sqrt{2}}\left[\beta_{00}^{(ij)} - \beta_{10}^{(ij)}\right].$$

Then (*) becomes

$$\frac{1}{2\sqrt{2}}\Big[\,|0\rangle^{(1)}(\beta_{00}^{(23)} + \beta_{10}^{(23)})\,|0\rangle^{(4)} + |0\rangle^{(1)}(\beta_{01}^{(23)} + \beta_{11}^{(23)})\,|1\rangle^{(4)}$$
$$+\,|1\rangle^{(1)}(\beta_{01}^{(23)} - \beta_{11}^{(23)})\,|0\rangle^{(4)} + |1\rangle^{(1)}(\beta_{00}^{(23)} - \beta_{10}^{(23)})\,|1\rangle^{(4)}\Big].$$

Measurement along the Bell basis at (23) yields one of the four possibilities

$$(1/\sqrt{2})\Big[\,|0\rangle^{(1)}\beta_{00}^{(23)}\,|0\rangle^{(4)} + |1\rangle^{(1)}\beta_{00}^{(23)}\,|1\rangle^{(4)}\Big] = \beta_{00}^{(23)} \otimes \beta_{00}^{(14)},$$

$$(1/\sqrt{2})\Big[\,|0\rangle^{(1)}\beta_{01}^{(23)}\,|1\rangle^{(4)} + |1\rangle^{(1)}\beta_{01}^{(23)}\,|0\rangle^{(4)}\Big] = \beta_{01}^{(23)} \otimes \beta_{01}^{(14)},$$

$$(1/\sqrt{2})\Big[\,|0\rangle^{(1)}\beta_{10}^{(23)}\,|0\rangle^{(4)} - |1\rangle^{(1)}\beta_{10}^{(23)}\,|1\rangle^{(4)}\Big] = \beta_{10}^{(23)} \otimes \beta_{10}^{(14)},$$

$$(1/\sqrt{2})\Big[\,|0\rangle^{(1)}\beta_{11}^{(23)}\,|1\rangle^{(4)} - |1\rangle^{(1)}\beta_{11}^{(23)}\,|0\rangle^{(4)}\Big] = \beta_{11}^{(23)} \otimes \beta_{11}^{(14)}.$$

Entanglement swapping: perfect and imperfect cases

Alice and Charlie may then do quantum teleportation using the (14) states. Any of the four Bell basis states may be used for teleportation.

Since the measurement outcome at (23) specifies the states at (1) and (4), one could apply quantum gates to put $\beta_{k\ell}^{(14)}$ into the state $\beta_{00}^{(14)}$. However, this would require non-local quantum operations, and the paradigm under consideration is LOCC.

In density-matrix terminology, one says that after entanglement swapping, the (14) state is *mixed*: it has a 4-point classical probability distribution.

* * *

As with quantum teleportation, this can again be done if the A-B and B-C links start off non-maximally entangled. It is shown in Perseguers et al., section III, that the average SCP p does not change.

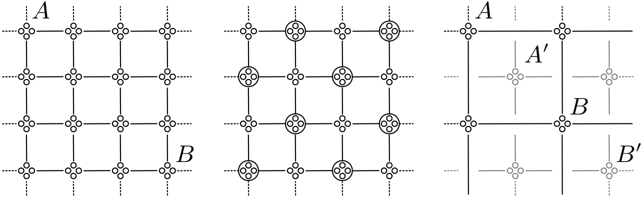
Review of the 2D square lattice

Quantum communication on the 2D square lattice; doubling

One may form a 1D chain of links. The probability of successful end-to-end communication over N links is p^N, which goes to zero in the infinite limit. One may instead leverage the well-known results of percolation to attempt to achieve higher teleportation probability on a 2D lattice. Perseguers et al. consider many lattice geometries; I confine my discussion to the square lattice.

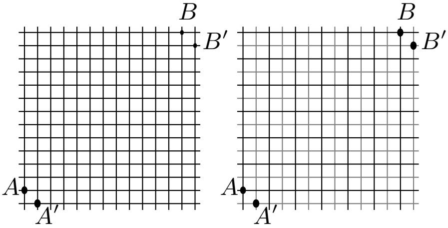
On the left is a square lattice formed of quantum-teleportation links. One may ask for the probability of communicating a qubit ψ (not shown) from point A to point B.

In the middle figure, we isolate Bob nodes and perform entanglement swapping twice per circle. The Bob nodes are discarded; what remains, in the right-hand figure, is a doubled lattice.



Doubling the 2D square lattice

In both cases, suppose that A is far from B, as are A' and B'. On the other hand, A and A', as well as B and B', occupy adjacent corners of a square. One may communicate along the black lattice from point A to point B, or along the grey lattice from point A' to point B'. Zoom out for a clearer look:



Recall that the percolation probability p is the same for the original lattice as for each of the doubled lattices.

Question: Which technique gives higher end-to-end teleportation probability — the original lattice or the doubled lattice?

Doubling the 2D square lattice

For the doubled lattice: If $p > p_c = 0.5$, there are infinite clusters $\mathcal{C}$, $\mathcal{C}'$ (black and grey, respectively) with probability 1. Successful communication from A to B requires $A, B \in \mathcal{C}$. These two events are (asymptotically) independent, so we have

$$P(A \in \mathcal{C}) = \theta(p), \qquad P(B \in \mathcal{C}) = \theta(p), \qquad P(A, B \in \mathcal{C}) = \theta^2(p).$$

Likewise, $P(A', B' \in \mathcal{C}') = \theta^2(p)$.

Taking advantage of both lattices, we can communicate from A's area to B's area if either path is open. We want to find $P(A, B \in \mathcal{C} \text{ or } A', B' \in \mathcal{C}')$.

Note that if events U and V are independent, $P(U \cup V)$ does not factor but $P(U \cap V)$ does. The inclusion-exclusion formula allows us to replace a union with an alternating sum of intersections, which factor. We find

$$\begin{aligned}
&P(A, B \in \mathcal{C} \text{ or } A', B' \in \mathcal{C}') \\
&= P(A, B \in \mathcal{C}) + P(A', B' \in \mathcal{C}') - P(A, B \in \mathcal{C} \text{ and } A', B' \in \mathcal{C}') \\
&= 2\theta^2(p) - \theta^4(p) \ = \theta^2(p)(2 - \theta^2(p)).
\end{aligned}$$

Doubling the 2D square lattice

For the non-doubled lattice, by comparison, there is a single infinite cluster $\mathcal{C}$. We want

$$P(A, B \in \mathcal{C} \ \text{ or } \ A, B' \in \mathcal{C} \ \text{ or } \ A', B \in \mathcal{C} \ \text{ or } \ A', B' \in \mathcal{C}).$$

Perseguers et al. claim (but omit the proof) that this is asymptotically $\pi^2(p)$ where

$$\begin{aligned} \pi(p) &= P(A \text{ or } A' \in \mathcal{C}) \\ &= P(A \in \mathcal{C}) + P(A' \in \mathcal{C}) - P(A, A' \in \mathcal{C}) \\ &= 2\theta(p) - \rho(p) \end{aligned}$$

where the second line again follows from inclusion-exclusion. In the third line, we have used the notation

$$\rho(p) = P(A, A' \in \mathcal{C}).$$

Proof of claim

Use inclusion-exclusion and the shorthand $P(AB) := P(A, B \in \mathcal{C})$, etc.:

$$\begin{aligned}
&P(AB \text{ or } AB' \text{ or } A'B \text{ or } A'B') \\
&= P(AB) + P(A'B) + P(AB') + P(A'B') \\
&- P(ABB') - P(AA'B) - P(AA'BB') - P(AA'BB') - P(AA'B') - P(A'BB') \\
&+ 4P(AA'BB') - P(AA'BB')
\end{aligned}$$

$$\begin{aligned}
&= P(AB) + P(A'B) + P(AB') + P(A'B') \\
&- P(ABB') - P(AA'B) - P(AA'B') - P(A'BB') + P(AA'BB').
\end{aligned}$$

Factor out asymptotically independent events:

$$\begin{aligned}
&P(AB \text{ or } AB' \text{ or } A'B \text{ or } A'B') \\
&= P(AB) + P(A'B) + P(AB') + P(A'B') \\
&- P(A)P(BB') - P(B)P(AA') - P(B')P(AA') - P(A')P(BB') + P(AA')P(BB') \\
&= 4\theta^2 - 4\theta\rho + \rho^2 = (2\theta - \rho)^2.
\end{aligned}$$

Comparison

To estimate $\pi^2(p)$, Perseguers et al. use the FKG inequality, namely, that increasing events are positively correlated. Thus

$$\rho(p) = P(A, A' \in \mathcal{C}) \geq P(A \in \mathcal{C}) P(A \circ\!\!-\!\!\circ A') = \theta(p)\tau(p)$$

where $\tau(p) := P(A \circ\!\!-\!\!\circ A')$ is the probability that A and A' are connected. This probability is clearly bounded below by p^2, but sharply: there may be circuitous paths connecting opposite edges of a corner.

In summary, the probability of successful communication on the doubled lattice is

$$P_{\text{double}} = \theta^2(p)(2 - \theta^2(p)),$$

versus on the non-doubled lattice

$$P_{\text{single}} = \pi^2(p) = (2\theta(p) - \rho(p))^2 \leq \theta^2(p)(2 - \tau(p))^2.$$

The doubled lattice is better if $P_{\text{double}} > P_{\text{single}}$, i.e. if

$$(2 - \tau(p))^2 \leq 2 - \theta^2(p).$$

At p_c, where $\theta = 0$, it suffices to show $\tau \geq 2 - \sqrt{2}$. This was done numerically.

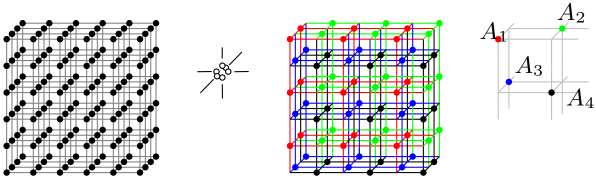
Quadrupling the 3D rectangular lattice

Quadrupling the 3D rectangular lattice

The first part of the figure shows the non-quadrupled lattice. The second part of the figure shows that each node actually has 6 qubits, although this detail is omitted from the rest of the figure for simplicity.

The third part shows the quadrupled lattice. In a manner analogous to the 2D case, center nodes do measurements onto the Bell basis and Bob themselves out of pariticipation. Four interlocking lattices — red, green, blue, and black — remain.

The fourth part shows the labeling of A_1, A_2, A_3, and A_4 which are analogs of A and A' in the 2D case.



As before, we ask whether successful communication on the quadrupled lattice is more likely than on the non-quadrupled lattice.

Quadrupling the 3D rectangular lattice

For the quadrupled lattice: If $p > p_c \approx 0.2488126$, there are infinite clusters $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ (red, green, blue, and black, respectively) with probability 1. Successful communication from A_i to B_i requires $A_i, B_i \in \mathcal{C}_i$ for $i = 1, 2, 3, 4$. These two events are (asymptotically) independent, so we have

$$P(A_i \in \mathcal{C}_i) = \theta(p), \qquad P(B_i \in \mathcal{C}_i) = \theta(p), \qquad P(A_i, B_i \in \mathcal{C}_i) = \theta^2(p).$$

Taking advantage of all four lattices, we can communicate from A_1's area to B_1's area if any of the four paths are open. Using inclusion-exclusion, we find

$$\begin{aligned}
P\left(\cup_{i=1}^4 (A_i, B_i \in \mathcal{C}_i)\right) &= \sum_{i=1}^4 P(A_i, B_i \in \mathcal{C}_i) - \sum_i \sum_{j \neq i} P(A_i, B_i \in \mathcal{C}_i \text{ and } A_j, B_j \in \mathcal{C}_j) \\
&+ \sum_i \sum_{j \neq i} \sum_{k \neq j} P(A_i, B_i \in \mathcal{C}_i \text{ and } A_j, B_j \in \mathcal{C}_j \text{ and } A_k, B_k \in \mathcal{C}_k) \\
&- P\left(\cap_{i=1}^4 (A_i, B_i \in \mathcal{C}_i)\right) \\
&= 4\theta^2(p) - 6\theta^4(p) + 4\theta^6(p) - \theta^8(p) \\
&= \theta^2(p)(4 - 6\theta^2(p) + 4\theta^4(p) - \theta^6(p)).
\end{aligned}$$

Quadrupling the 3D rectangular lattice

For the non-quadrupled lattice, there is a single infinite cluster $\mathcal{C}$. One can show that

$$P\left(\cup_{i=1}^{4} \cup_{j=1}^{4} (A_i, B_j \in \mathcal{C})\right)$$

reduces, as in the 2D case, asymptotically to $\sigma^2(p)$ where

$$\sigma(p) := P(\cup_{i=1}^{4} A_i \in \mathcal{C}).$$

Proof: See the next slide.

As shown on slide 20, Perseguers et al. then used inclusion-exclusion and FKG to split the long-scale event $A, A' \in \mathcal{C}$ into the event $A \in \mathcal{C}$, with well-known probability $\theta(p)$, and the short-scale event $A \circ\!\!-\!\!\circ A'$.

I found that this was not productive for 3D. Inclusion-exclusion expansion of $\sigma(p)$ gives four terms: positive, negative, positive, and negative. For the second and fourth terms we can use FKG; for the third term we need not a *lower* bound but an *upper* bound. Here I was stuck at the end of the spring semester and didn't see hope of further progress.

John LaPeyre pointed out to me recently, though, that $\sigma(p)$ can be directly attacked numerically. I don't need to do the inclusion-exclusion expansion at all. I simply need to see if $\sigma^2(p)$ is greater than the polynomial in $\theta(p)$ on the previous slide.

Proof of claim

Apply inclusion-exclusion to the outer union. Let $E_i = \cup_{m=1}^4 (A_i, B_m \in \mathcal{C})$. Then

$$P(\cup_{i=1}^4 E_i) = \sum_i P(E_i) - \sum_{i \neq j} P(E_i, E_j) + \sum_{i \neq j \neq k} P(E_i, E_j, E_k) - P(E_1, E_2, E_3, E_4).$$

Now

$$P(E_i) = P\left(\cup_{m=1}^4 (A_i, B_m \in \mathcal{C})\right) = P\left(A_i \in \mathcal{C}; \cup_{m=1}^4 (B_m \in \mathcal{C})\right),$$
$$P(E_i, E_j) = P\left(A_i, A_j \in \mathcal{C}; \cup_{m=1}^4 (B_m \in \mathcal{C})\right)$$

and similarly for $P(E_i, E_j, E_k)$ and $P(E_1, E_2, E_3, E_4)$. Also note that $A_i \in \mathcal{C}$ is asymptotically independent of $B_m \in \mathcal{C}$. So

$$\begin{aligned} P(\cup_{i=1}^4 E_i) = & \left[\sum_i P(A_i \in \mathcal{C}) - \sum_{j \neq i} P(A_i, A_j \in \mathcal{C}) \right. \\ & \left. + \sum_{k \neq j \neq i} P(A_i, A_j, A_k \in \mathcal{C}) - P(A_1, A_2, A_3, A_4 \in \mathcal{C}) \right] \left[\cup_{m=1}^4 (B_m \in \mathcal{C}) \right] \\ = & \left[\cup_{i=1}^4 (A_i \in \mathcal{C}) \right] \left[\cup_{m=1}^4 (B_m \in \mathcal{C}) \right] = \sigma^2(p). \end{aligned}$$

Monte Carlo simulations

Monte Carlo simulations

Overview:

- For $M = 20, 25, 30, 35, 40, 45, \ldots$ as far as patience and CPU time hold out, and for various values of p above p_c, estimate

$$\sigma_M(p) := P_M\left(\cup_{i=1}^{4}(A_i \in \mathcal{C})\right)$$

  for $M \times M \times M$ lattices. (Note that this is now strictly a percolation question: quantum information is out of the picture.)
- For each fixed p, use finite-size scaling to extrapolate $\sigma(p) = \lim_{M\to\infty} \sigma_M(p)$.
- Find $\lim_{p \searrow p_c} \sigma(p)$. One expects this to follow a power law, as $\theta(p)$ does.

It will be helpful to do this also for $\tau(p)$ — $P(A \circ\!\!-\!\!\circ A')$ — to recover the 2D results from Perseguers et al., and also to estimate $\theta(p)$ — $P(A \in \mathcal{C})$ — for the 2D and 3D cases. These comparisons against known results provide a sanity check for my finite-size scaling.

Monte Carlo simulations for fixed M and p

The algorithms are simple.

To estimate a single P_M value for one p, do N trials detecting the event $\cup_{i=1}^4 (A_i \in \mathcal{C})$. Average these over the N trials to estimate P_M of that event. When choosing N, recall that the sample mean tends centrally toward a normal distribution and that the normal's standard deviation goes as $1/\sqrt{N}$. (I.e. to get another decimal place in the estimate of $P_M(E)$ for some event E, one needs to run 100 times as many experiments.)

For each trial:

- Populate the bonds of the lattice. Each is open with probability p.
- When computing $P_M(A \multimap\!\circ\, A')$, do a cluster walk (described below).
- When computing $\theta_M(p)$ or $\sigma_M(p)$, mark all clusters and identify the largest one (as described below). Once the largest cluster is marked, it is easy to find if one point (for θ) or any of a specified four (for σ) are in that cluster.

Cluster walking

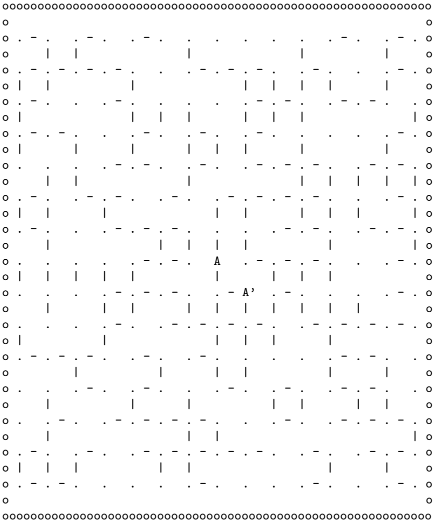
To see if $A \circ\!\!-\!\!\circ A'$, a naive algorithm almost works:

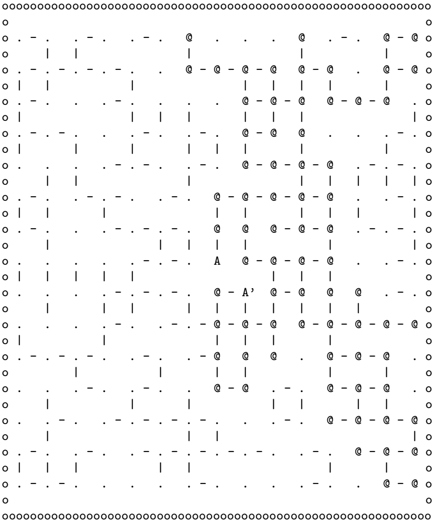
- Start at site A.
- Make a list of the 0 to 4 nearest-neighbor sites which are connected to A by an open bond.
- If any of those sites is A', then $A \circ\!\!-\!\!\circ A'$. Stop.
- Otherwise, repeat this process (by recursively calling the subroutine) for each of the neighbors.
- Once the recursions are complete with no more unmarked neighbors to visit, A is not connected to A'. Stop.

Problem: you can chase around in a circle indefinitely whenever there is a loop in the bond graph.

Solution: Make a matrix of site marks, all initialized to zero. Mark each site as you visit it. When recursively calling the subroutine, recurse only into non-visited sites. Infinite recursion successfully avoided.

Cluster walk with $M = 14$ and $p = 0.51$: before and after





Cluster marking and sizing

Cluster marking:

- Again keep a matrix of site marks, now serving as cluster numbers, all initially set to zero.
- Set cluster number = 1.
- For each site A:
  - If A's cluster number is non-zero (site A has already been visited), continue to the next site.
  - In the site-marks matrix, mark A with the current cluster number.
  - For each bonded neighbor of A, recursively call the subroutine.
  - After the recursion, increment the cluster number by 1.

Cluster sizing:

- Walk through the sites of the lattice, counting the size of each cluster.
- Remember the cluster number of the largest cluster. Call this $\mathcal{C}$.

Lattice before and after cluster numbering: $M = 14, p = 0.6$

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o                                                           o
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o 2   2 - 2   2 - 2   2 - 2 - 2 - 2   2   5   4 - 4 - 4 - 4 o
o |   |   |   |               |   |   |                     o
o 2 - 2   2 - 2 - 2 - 2   2 - 2   2   2   2 - 2 - 2   2 - 2 o
o     |   |   |   |   |           |   |   |       |   |   | o
o 6   2 - 2 - 2 - 2   2   2 - 2 - 2 - 2   2 - 2 - 2 - 2 - 2 o
o |       |   |                   |   |   |           |     o
o 6   2 - 2   2   2 - 2   2 - 2 - 2 - 2   2   2 - 2 - 2 - 2 o
o     |   |   |   |   |   |   |   |   |   |   |       |     o
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o             |           |   |   |   |   |   |       |   | o
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o |       |       |   |   |   |   |   |   |   |   |         o
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Finite-size scaling

Finite-size scaling

Finite-size scaling analysis is in progress. Thoughts gleaned from Kennedy and LaPeyre:

- M and ξ (correlation length) are both length scales.
- For p comfortably above or below p_c, M passes ξ quickly and infinite values $\sigma(p)$ are obtained quickly.
- For p near p_c, one must somehow extrapolate $\sigma_M(p)$ to $\sigma(p)$.
- ξ can be estimated numerically.
- Kennedy conjectures $\sigma_M(p) = \sigma(p)F(M/\xi)$, for some F.
- See perhaps Stauffer's text.
- Wehr: Do the comparison just for one or more p's off p_c. This is more easily achieved and might be sufficiently newsworthy.

See the next slide for some data, obtained as follows.

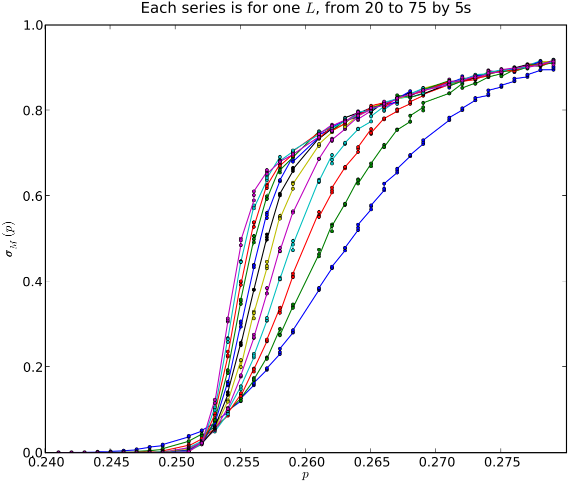
For p from 0.241 up to 0.279 in steps of 0.001:

  For M from 20 to 75 in steps of 5:

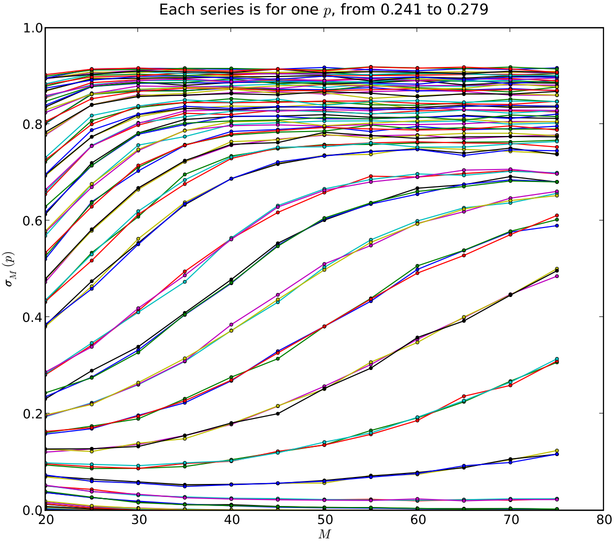
    For three trials:

      Plot $\sigma_M(p)$.

Finite-size scaling: $\sigma_M(p)$ vs. p



Finite-size scaling: $\sigma_M(p)$ vs. M



Conclusions

- All finite-lattice questions raised here are easily solved by simulation.
- Finite-size-scaling analysis needs to be completed — *after* my oral exam!
- Thank you for your time!