Exam #2 Solutions · Monday November 20, 2006

MATH 111 · Section 7 · Fall 2006 Name ______________________________

The solutions I've given here include my explanations and thus are wordier than I would expect your responses to be. Only problem 13 is a writing problem; for the rest, computational answers suffice.

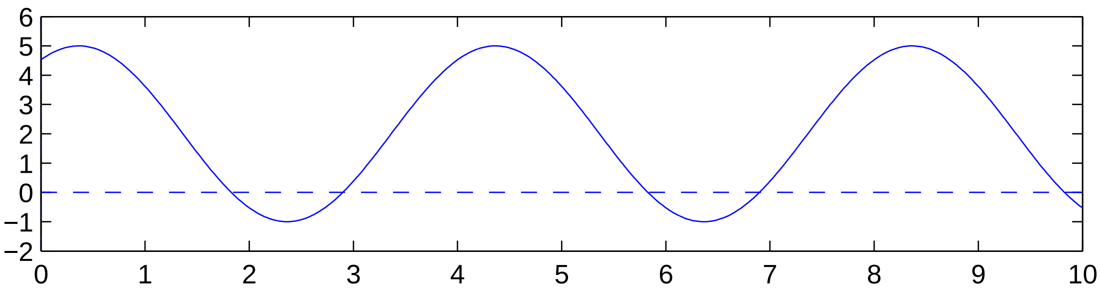
Problem 1. Given the function $y = \frac{1}{3}\sin(4x+1)$, find the average value and period.

Solution: The average value is what's added to the sin term, which is zero. For the period, solve the inequality

$$\begin{aligned} 0 &\le 4x+1 &&\le 2\pi \\ -1 &\le 4x &&\le 2\pi - 1 \\ -\frac{1}{4} &\le x &&\le \frac{2\pi}{4} - \frac{1}{4}. \end{aligned}$$

The period is the right-hand side minus the left-hand side, which is $\pi/2$.

Problem 2. Given the following graph, find the maximum value, minimum value, and amplitude.

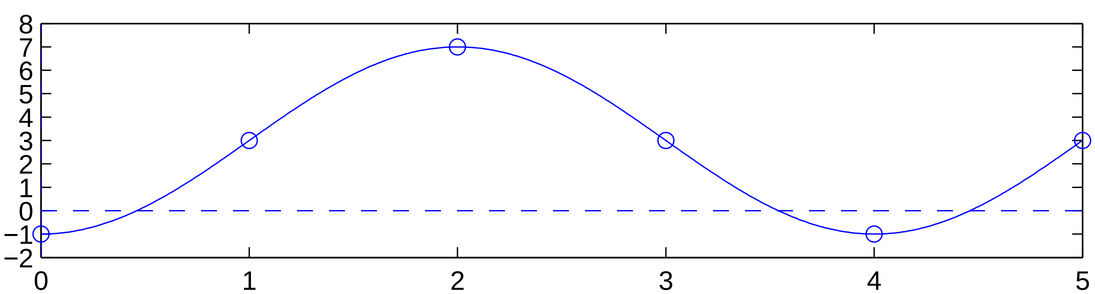


Solution: Read off the values max = 5, min = -1, and amplitude = 3. Or, amplitude is $(\max - \min)/2 = (5+1)/2 = 3$.

Problem 3. Write an equation for a sine function which passes through all the points in the following data table:

x	0	1	2	3	4	5
y	-1	3	7	3	-1	3

Solution: First graph the function:



Now read off the amplitude, average value, period, and phase shift from the graph and write down

$$\begin{aligned} y &= \text{avg. val.} + \text{amplitude} \cdot \sin\left(\frac{2\pi}{\text{period}}(x - \text{phase shift})\right) \\ y &= 3 + 4\sin\left(\frac{2\pi}{4}(x-1)\right) = 3 + 4\sin\left(\frac{\pi}{2}(x-1)\right). \end{aligned}$$

Problem 4. Which of the following correctly represents $\cos(-\theta) + \sin(-\theta)$?

(A) $\cos(\theta) + \sin(\theta)$ (B) $\cos(\theta) - \sin(\theta)$ (C) $-\cos(\theta) + \sin(\theta)$ (D) $-\cos(\theta) - \sin(\theta)$
(E) None of these.

Solution: Since cosine is an even function and sine is an odd function, we have, for all θ, $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$. So, the answer is B.

Problem 5. Verify the following trigonometric identity. (Hint: use a sum identity and the Pythagorean identity.)

$$\cos(2x) = 1 - 2\sin^2(x).$$

Solution: The sum identity for cosine gives us

$$\begin{aligned} \cos(2x) &= \cos(x + x) \\ &= \cos(x)\cos(x) - \sin(x)\sin(x) \\ &= \cos^2(x) - \sin^2(x). \end{aligned}$$

You're asked to get the right-hand side looking like $1 - 2\sin^2(x)$, but there is a $\cos^2(x)$ term in there to be gotten rid of. Fortunately, we know from the Pythagorean identity that

$$\cos^2(x) + \sin^2(x) = 1.$$

Using this, we can put

$$\cos^2(x) = 1 - \sin^2(x)$$

into the above, picking up from where we left off, to obtain

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= (1 - \sin^2(x)) - \sin^2(x) \\ &= 1 - 2\sin^2(x). \end{aligned}$$

Problem 6. Determine the exact value of $\sin(7\pi/12)$ using sum and/or difference identities.

Solution: This is perhaps easier in degrees, perhaps not. Using degrees, we have

$$\frac{7\pi}{12} = \frac{7\pi\, 180^\circ}{12\pi} = 7 \cdot 15^\circ = 105^\circ.$$

Now, $105^\circ = 60^\circ + 45^\circ$ so we can take advantage of known values for sine and cosine at 60° and 45°, along with the sum identity for sine:

$$\begin{aligned} \sin(60^\circ + 45^\circ) &= \sin(60^\circ)\cos(45^\circ) + \cos(60^\circ)\sin(45^\circ) \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}. \end{aligned}$$

Using radians, we have

$$\frac{7\pi}{12} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$$

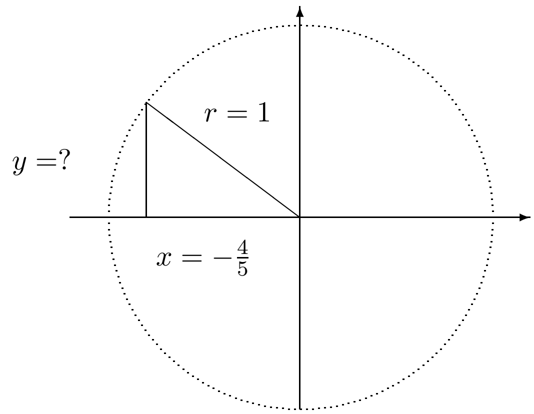
so we can take advantage of known values for sine and cosine at $\pi/3$ and $\pi/4$, along with the sum identity for sine:

$$\begin{aligned} \sin(\pi/3 + \pi/4) &= \sin(\pi/3)\cos(\pi/4) + \cos(\pi/3)\sin(\pi/4) \\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}. \end{aligned}$$

Remember that switching between radians and degrees changes the units of the *input* to the trigonometric functions, but doesn't change the *output*. So, we get the same answer either way.

Problem 7. If $\cos(\theta) = -4/5$ and $\sin(\theta) > 0$, find $\tan(\theta)$.

Solution: Since $\cos(\theta) = x/r$ is negative, θ is in quadrant II or III; since $\sin(\theta) = y/r$ is positive, θ must be in quadrant II. Now we can make a cartoon of what we're looking for, to make it easier to do the math. We put the angle θ in standard position with a point (x, y) on the terminal edge and also on the unit circle:



Since $x^2 + y^2 = 1$ on the unit circle — which is the same as $\cos^2(\theta) + \sin^2(\theta) = 1$ — we can write

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \\ &= 1 - \left(\frac{-4}{5}\right)^2 = 1 - \frac{16}{25} = \frac{9}{25} \\ y &= \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}. \end{aligned}$$

Since we are given $\sin(\theta) > 0$, we know to pick $y = 3/5$. Then

$$\tan(\theta) = \frac{y}{x} = \frac{3/5}{-4/5} = \frac{-3}{4}.$$

Problem 8. Which of the following is not an identity?

(A)	$\sin^2(x) + \cos^2(x) = 1$	(B)	$\cos(x + y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
(C)	$\sin(2x) = 2\sin(x)\cos(x)$	(D)	$\cos(\pi/2 - x) = \sin(x)$
(E)	These are all identities.		

Solution: B is not an identity. The correct version of the sum identity for cosine is

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Answer A is the Pythagorean identity; C is the double-angle identity for sine, which follows immediately from the sum identity for sine; D is a cofunction identity.

Problem 9. Find all solutions for $0 \le x < 2\pi$ (or, if you prefer, do it in degrees for $0° \le x < 360°$):

$$(\sin(x) + 1/2)(\sin(x) - 1/2) = 0.$$

Solution: Whenever the product of two (or more) real numbers is zero, at least one of them is zero. So, we have two cases:

- If the right-hand factor is zero, then $\sin(x) = 1/2$. There are two points on the unit circle with height $1/2$: they are $30°$ and $150°$.
- If the left-hand factor is zero, then $\sin(x) = -1/2$. There are two points on the unit circle with height $-1/2$: they are $210°$ and $330°$. (If you're like me, you'd more naturally think of $330°$ as $-30°$. But, the problem asked for solutions between $0°$ and $360°$.)

So, we have four solutions: $30°$, $150°$, $210°$, and $330°$. In radians, these are $\pi/6$, $5\pi/6$, $7\pi/6$, and $11\pi/6$.

Problem 10. Solve for x:

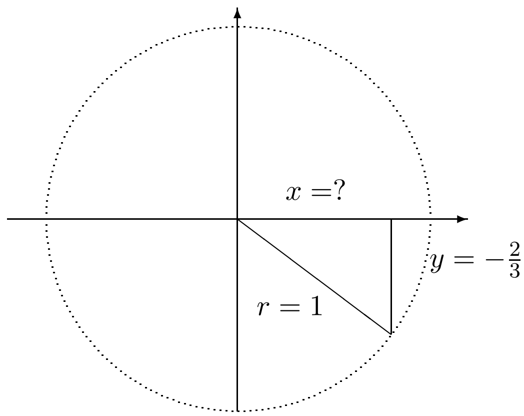
$$y = 5\ \cos^{-1}(2x) + 3.$$

Solution:

$$\begin{aligned} y &= 5\ \cos^{-1}(2x) + 3 \\ y - 3 &= 5\ \cos^{-1}(2x) \\ \frac{y-3}{5} &= \cos^{-1}(2x) \\ \cos\left(\frac{y-3}{5}\right) &= 2x \\ \frac{1}{2}\ \cos\left(\frac{y-3}{5}\right) &= x. \end{aligned}$$

Problem 11. Find the exact value of $\cos(\sin^{-1}(-2/3))$.

Solution: Remember that trig functions take angles to numbers, while their inverse functions take numbers to angles. So, $\sin^{-1}(-2/3)$ is an angle. Which angle is it? The $\sin^{-1}$ function has a range of $-90°$ to $90°$; $-2/3$ is negative so we are looking for an angle θ with negative height, where $\sin(\theta) = -2/3$. We can draw a cartoon to help us get the math right:



On the unit circle, $r = 1$ and so $\cos(\theta) = x/r = x/1 = x$. We can use the Pythagorean theorem, $x^2 + y^2 = 1$, to find

$$\begin{aligned} x^2 + \left(\frac{-2}{3}\right)^2 &= 1 \\ x^2 + \frac{4}{9} &= 1 \\ x^2 &= 1 - \frac{4}{9} = \frac{5}{9} \\ x &= \pm\frac{\sqrt{5}}{3}. \end{aligned}$$

All angles given back by the sine function are between $-90°$ and $90°$ and so are in quadrants I and IV, where x is non-negative. So, we choose the positive square root to get $x = \frac{\sqrt{5}}{3}$.

Problem 12. Find the exact value of $\arcsin(\sin(135°))$.

Solution: From chapter 1, $\sin(135°)$ is one of our well-known values, namely, $\sqrt{2}/2$. Now remember that trig functions map angles to numbers, while their inverse functions map numbers to angles. There are, of course, *many* angles whose sine is $\sqrt{2}/2$, and functions (in order to be functions) can have only one output value. We define the inverse sine function to map numbers from -1 to 1 to angles between $-90°$ and $90°$. So, $\arcsin(\sqrt{2}/2) = 45°$.

Problem 13. Writing question: Why is it that $\tan^{-1}(2)$ is defined, while $\sin^{-1}(2)$ and $\cos^{-1}(2)$ are undefined?

Solution: The domain of the tangent function is $(-\infty, +\infty)$ while the domain of the sine and cosine functions is $[-1, 1]$; 2 is in the former but not the latter.

The above sentence is an acceptable answer, of acceptable length. For more explanation of why that, in turn, is true, though:

- Remember that trig functions map angles to numbers, and their inverse functions map numbers to angles.

- Also remember that when we draw an angle in standard position, then pick a point (x, y) which is on the terminal edge of the angle and also on the unit circle, the y coordinate (the height) of that point is the sine of the angle. Likewise, the x coordinate of that point is the cosine of the angle. The tangent of the angle is y/x which is the slope of the hypotenuse. The y and x coordinates are on the unit circle, so they're confined between -1 and 1. The slope, on the other hand, can get as steep as we want.
- Last, remember that when we compute $\sin^{-1}$ of a number, we're asking: *What's our favorite angle whose sine is that number?* Similarly for the $\cos^{-1}$ and $\tan^{-1}$ functions.

So, since sine and cosine don't have value 2 for any angle, we won't be able to find an angle with sine or cosine equal to 2. On the other hand, since the tangent function takes on all real values, we will be able to find an angle with tangent equal to 2.