Exam #3 solutions · Thursday, November 6, 2008

MATH 124 · Calculus I · Section 26 · Fall 2008

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, unless a problem specifically requests a verbal response.
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Problem 1. Consider the following table of values of a function f and its first two derivatives.

x	-3	-1	1	3	5
$f(x)$	3.0	4.0	5.0	8.0	9.0
$f'(x)$	1.0	0.0	0.5	1.0	0.3
$f''(x)$	-2.0	0.0	1.4	0.0	-0.4

Furthermore, f' and f'' have no zeroes other than the ones shown.
Part (a). Does f have a critical point at $x = -1$? Why or why not?
Solution: True: definition of critical point.
Part (b). Does f have a local maximum at $x = -1$? Why or why not?
Solution: False: f' is positive on either side, so f fails the first derivative test at $x = -1$. The second derivative test is inconclusive since $f''(-1)$ is 0.
Part (c). Does f have an inflection point at $x = 3$? Why or why not?
Solution: True: $f''(3) = 0$ and f'' changes sign at $x = 3$.
Problem 2. Consider the curve

$$\ln(xy) = 2x.$$

Part (a). Find dy/dx.
Solution: Differentiating implicitly, we have

$$\begin{aligned} \frac{d}{dx}(\ln(xy)) &= \frac{d}{dx}(2x) \\ \frac{1}{xy}\left(y + x\frac{dy}{dx}\right) &= 2 \\ y + x\frac{dy}{dx} &= 2xy \\ x\frac{dy}{dx} &= 2xy - y \\ \frac{dy}{dx} &= \frac{y(2x-1)}{x}. \end{aligned}$$

Part (b). Find an equation for the tangent line the curve at the point $(x, y) = (1, e^2)$.
Solution: Plugging $x = 1$ and $y = e^2$ into the above, we have

$$\left.\frac{dy}{dx}\right|_{x=1,y=e^2} = \frac{e^2(2\cdot 1 - 1)}{1} = e^2.$$

This is the m. The a and b are 1 and e^2. So we have

$$\begin{aligned} y &= m(x-a) + b \\ &= e^2(x-1) + e^2 \\ &= e^2 x. \end{aligned}$$

Note: We can also differentiate this explicitly if we like. First solve for y:

$$\begin{aligned} \ln(xy) &= 2x \\ xy &= e^{2x} \\ y &= \frac{e^{2x}}{x} \\ \frac{dy}{dx} &= \frac{2xe^{2x} - e^{2x}}{x^2} = \frac{e^{2x}(2x-1)}{x^2} \\ \left.\frac{dy}{dx}\right|_{x=1, y=e^2} &= \frac{e^2(2-1)}{1^2} = e^2. \end{aligned}$$

The tangent-line calculation is the same.

Problem 3.

Part (a). Find B so that

$$G(x) = B2^x + 2^{-x}$$

has a critical point at $x = -1$.

Solution: A critical point is where $G'(x) = 0$ or is undefined. This is an everywhere differentiable function, so we need to solve $G'(x) = 0$.

$$\begin{aligned} G'(x) &= B\ln(2)2^x - \ln(2)2^{-x} \\ &= \ln(2)\left(B2^x - 2^{-x}\right). \end{aligned}$$

Solving for B in $G'(-1) = 0$, we have

$$\begin{aligned} \ln(2)\left(B2^{-1} - 2^1\right) &= 0 \\ B2^{-1} &= 2^1 \\ \frac{1}{2}B &= 2 \\ B &= 4. \end{aligned}$$

Part (b). Is this critical point a minimum, maximum, or neither? Explain your reasoning.

Solution: This is a minimum, as can be read off the graph. Alternatively, note that the second derivative is $G''(x) = \ln(2)^2(B2^x + 2^{-x})$. This is positive for all x, and in particular for $x = -1$. So, by the second-derivative test, this critical point is a local (in fact, global) minimum.

Problem 4. For each of the following, does the limit exist? If so, what is it, and why? If not, why not?

Part (a). $\lim_{t\to 0} \frac{1-\cosh t}{t^2}$

Solution: Since $\cosh(0) = 1$, this is a $0/0$ situation and l'Hôpital's rule applies. Differentiating top and bottom, we have

$$\lim_{t\to 0} \frac{1-\cosh t}{t^2} = \lim_{t\to 0} \frac{-\sinh t}{2t}.$$

This is still a $0/0$ situation, so we need to use l'Hôpital's rule again:

$$\lim_{t\to 0} \frac{-\sinh t}{2t} = \lim_{t\to 0} \frac{-\cosh t}{2}.$$

Now we can set $t = 0$ to obtain $-1/2$. (This also matches what you can see if you graph $(1-\cosh(t))/t^2$ on your calculator.)

Part (b). $\lim_{z\to 0} \frac{3^z}{z^3}$

Solution: The numerator goes to 1 as $z \to 0$; the denominator goes to zero as $z \to 0$. Thus, the limit does not exist. (This is neither a $0/0$ case nor an ∞/∞ case, so l'Hôpital's rule does not apply.)

Problem 5. The east and west sides of a rectangular enclosure cost \$50 per meter; the north and south sides cost \$80 per meter. Find the dimensions of the enclosure with least cost enclosing an area of 1000 square meters.
Solution: Let x be the length of the north and south sides; Let y be the length of the east and west sides. Writing down the given information, we have

$$\begin{aligned} xy &= 1000 \\ C &= 160x + 100y. \end{aligned}$$

We need to optimize the cost C, which is currently in terms of two variables. Using the area expression to eliminate a variable we get

$$\begin{aligned} y &= \frac{1000}{x} \\ C &= 160x + \frac{100000}{x}. \end{aligned}$$

Extrema occur at boundary points and critical points. When $x = 0$, there is no area; when x gets big (so y goes to 0) there is no area. So, it suffices to look at critical points of C. We have

$$\begin{aligned} \frac{dC}{dx} &= 160 - \frac{100000}{x^2}. \\ 160 - \frac{100000}{x^2} &= 0 \\ \frac{100000}{x^2} &= 160 \\ x^2 &= \frac{100000}{160} = \frac{10000}{16} \\ x &= \sqrt{\frac{10000}{16}} = \frac{100}{4} = 25. \end{aligned}$$

Then

$$y = \frac{1000}{25} = 40.$$

Problem 6. The trajectory of an orbiting object is described by

$$r(1 + 0.2\cos(\theta)) = 10.$$

(The units of r are thousands of kilometers, or megameters if you like, and the units of θ are radians.) Find $d\theta/dt$ when $\theta = \pi/3$, given that $dr/dt = -20$ megameters per hour when $\theta = \pi/3$. Compute your answer to three decimal places.
Solution: First note, as I announced in class, that you don't need to know polar coordinates to do this problem.
Here the quantities r and θ both vary in time; these quantities are related by the above equation. Differentiating both sides with respect to time, we get

$$\begin{aligned} \frac{dr}{dt}(1 + 0.2\cos(\theta)) - 0.2r\sin(\theta)\frac{d\theta}{dt} &= 0 \\ \frac{dr}{dt}(1 + 0.2\cos(\theta)) &= 0.2r\sin(\theta)\frac{d\theta}{dt} \\ \frac{d\theta}{dt} &= \frac{\frac{dr}{dt}(1 + 0.2\cos(\theta))}{0.2r\sin(\theta)}. \end{aligned}$$

To evaluate this at $\theta = \pi/3$, we need dr/dt there, which is -20; $\sin(\theta)$ which is $\sqrt{3}/2 \approx 0.866$; $\cos(\theta)$ which is 0.5; and r. The latter is

$$r = \frac{10}{1 + 0.2\cos(\theta)} = \frac{10}{1 + 0.2 \cdot 0.5} \approx 9.091.$$

Then

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = \frac{-20(1+0.2\cdot 0.5)}{0.2\cdot 9.091\cdot 0.866}$$
$$\approx -13.972 \text{ radians/meter.}$$