Exam #4 solutions · Thursday, December 4, 2008

MATH 124 · Calculus I · Section 26 · Fall 2008

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, unless a problem specifically requests a verbal response.

John Kerl (`kerl at math dot arizona dot edu`).

Problem 1. Let

$$x(t) = 2t^3 - 15t^2 + 24t + 7$$
$$y(t) = t^2 + t + 1.$$

Find all values of t such that this curve has a vertical tangent line.
Solution: The tangent line is vertical when $dx/dt = 0$:

$$\begin{aligned} x(t) &= 2t^3 - 15t^2 + 24t + 7 \\ \frac{dx}{dt} &= 6t^2 - 30t + 24 \\ 6t^2 - 30t + 24 &= 0 \\ t^2 - 5t + 4 &= 0 \\ (t-1)(t-4) &= 0 \\ t &= 1, 4. \end{aligned}$$

Problem 2.
Part (a). Find the average value of

$$G(y) = \frac{A}{y} + By$$

over the interval $-4 \le y \le -1$, where A and B are positive constants.
Solution: The average value is

$$\frac{1}{b-a} \int_{y=a}^{y=b} G(y)\, dy$$

with $a = -4$ and $b = -1$. This is

$$\begin{aligned} \frac{1}{-1 - -4} \int_{y=-4}^{y=-1} \left(\frac{A}{y} + By \right) dy &= \frac{A}{3} \int_{y=-4}^{y=-1} \frac{1}{y}\, dy + \frac{B}{3} \int_{y=-4}^{y=-1} y\, dy \\ &= \frac{A}{3} \left[\ln(|y|)\right]_{y=-4}^{y=-1} + \frac{B}{3} \left[\frac{y^2}{2}\right]_{y=-4}^{y=-1} \\ &= \frac{A}{3} (\ln(1) - \ln(4)) + \frac{B}{3} \left(\frac{1}{2} - \frac{16}{2} \right) \\ &= -\frac{A\ln(4)}{3} - \frac{7.5B}{3} \\ &= -\frac{A\ln(4)}{3} - 2.5B. \end{aligned}$$

Part (b). If the units of y and $G(y)$ are inches and gallons, respectively, what are the units of the average value found in part (a)?
Solution: Intuitively, the average value of a gallons function should be gallons. Mechanically, the units of the integral are gallons times inches; the denominator $(-1 - -4)$ has units of inches. Thus,

$$\frac{1}{\text{inches}} \cdot (\text{gallons} \cdot \text{inches}) = \text{gallons}.$$

Problem 3. Suppose the rate of change of the price of a stock is $R(t)$ where t is measured in days since the start of the year and $R(t)$ is dollars per day.

Part (a). What are the units of $\int_{31}^{59} R(t)dt$?

Solution: The units of the integral are R units times t units. This is dollars per day times days, i.e. dollars.

Part (b). Give a practical interpretation of $\int_{31}^{59} R(t)dt$.

Solution: This is the change in the stock price between the 31st and 59th days of the year. (Note that Jan. 1 would be the 0th day.) You may also notice that days 31-59 are the month of February (in a non-leap year).

Part (c). If $R(t) < 0$ for $30 \leq t \leq 60$, can you tell what the sign of $\int_{31}^{59} R(t)dt$ is? If so, what is the sign? If not, why not?

Solution: If the integrand is negative throughout the interval of integration, then the integral's value will be negative: all the area will be below the horizontal axis.

Part (d). If $R(t) < 0$ for $30 \leq t < 40$ and $R(t) > 0$ for $40 < t \leq 60$, can you tell what the sign of $\int_{31}^{59} R(t)dt$ is? If so, what is the sign? If not, why not?

Solution: The falling-price period (9 days) is shorter than the rising-price period (19 days), so one might be tempted to say there was a net gain. However, we don't know anything about the magnitude of R — the area below the horizontal axis for days 31-40 might be more or less than the area above the horizontal axis for days 40-59. So, we can't determine the sign of the integral based on the information we have.

Problem 4. Find the exact area of the region between $f(x) = x^{2/3}$ and $g(x) = \sin(x)$ over the interval $1/8 \leq x \leq \pi$.

Solution: Graphing, we see that $x^{2/3}$ is on top of $\sin(x)$. So, we compute

$$\begin{aligned}
\int_{x=1/8}^{x=\pi} (f(x) - g(x))\,dx &= \int_{x=1/8}^{x=\pi} (x^{2/3} - \sin(x))\,dx \\
&= \left[\frac{3}{5}x^{5/3} + \cos(x)\right]_{x=1/8}^{x=\pi} \\
&= \left(\frac{3}{5}\pi^{5/3} + \cos(\pi)\right) - \left(\frac{3}{5}\left(\frac{1}{8}\right)^{5/3} + \cos(1/8)\right) \\
&= \frac{3}{5}\pi^{5/3} - 1 - \frac{3}{5}\frac{1}{32} - \cos(1/8) \\
&= \frac{3}{5}\pi^{5/3} - \cos(1/8) - 1 - \frac{3}{160} \\
&= \frac{3}{5}\pi^{5/3} - \cos(1/8) - \frac{163}{160}.
\end{aligned}$$

Problem 5. Label the following statements as correct (C) or incorrect (I):

Part (a). If $f'(x) < 0$ on an interval then $f(x)$ is decreasing on that interval.

Solution: This is correct.

Part (b). If $f'(x) = 0$ at a point then $f(x)$ has an inflection point at that point.

Solution: This is incorrect. Change "inflection point" to "critical point" and it would be correct. A critical point of $f(x)$ is, by defintion, a point where $f'(x)$ is zero or undefined.

Part (c). If $f'(x)$ has a maximum or minimum at a point then $f(x)$ has a critical point there.

Solution: This is incorrect. Change "critical point" to "inflection point" and it would be correct.

Problem 6. Let

$$\frac{dP}{dt} = t(1-t).$$

Part (a). Find a general solution for P.
Solution: Antidifferentiating both sides, we get

$$\begin{aligned} P &= \int t(1-t)\,dt \\ &= \int (t-t^2)\,dt \\ &= \frac{t^2}{2} - \frac{t^3}{3} + C. \end{aligned}$$

Part (b). Find a specific solution for P if $P(3) = 2$.
Solution:

$$\begin{aligned} P &= \frac{t^2}{2} - \frac{t^3}{3} + C \\ 2 &= \frac{3^2}{2} - \frac{3^3}{3} + C \\ 2 &= \frac{9}{2} - 9 + C \\ 2 &= -\frac{9}{2} + C \\ C &= 2 + \frac{9}{2} = \frac{13}{2} \\ P &= \frac{t^2}{2} - \frac{t^3}{3} + \frac{13}{2}. \end{aligned}$$

Problem 7. Let $F(x) = \int_{10}^{x} f(t)dt$, with the following known values for $f(t)$:

t	0	10	20	30	40
$f(t)$	-1.25	-1.2	-1.1	-0.9	-0.7

Estimate $F(20)$ and $F(30)$.
Solution: We have

$$F(20) = \int_{10}^{20} f(t)\,dt \quad \text{and} \quad F(30) = \int_{10}^{30} f(t)\,dt.$$

There are various ways to estimate this.

- Using left-hand sums, we have

  $$\int_{10}^{20} f(t)\,dt \approx 10 \cdot (-1.2) = -12$$

  and

  $$\int_{10}^{30} f(t)\,dt \approx 10 \cdot (-1.2) + 10 \cdot (-1.1) = -12 - 11 = -23.$$

- Using right-hand sums, we have

  $$\int_{10}^{20} f(t)\,dt \approx 10 \cdot (-1.1) = -11$$

  and

  $$\int_{10}^{30} f(t)\,dt \approx 10 \cdot (-1.1) + 10 \cdot (-0.9) = -11 - 9 = -20.$$

Problem 8.
Part (a). Find

$$\frac{d}{d\theta} \int_{\pi/2}^{\theta} \frac{\sin(x)}{x} dx.$$

Solution: When we see the derivative of an integral function, we should think of the Second Fundamental Theorem of Calculus:

$$\frac{d}{d\theta}\int_{\pi/2}^{\theta}\frac{\sin(x)}{x}\,dx = \frac{\sin(\theta)}{\theta}.$$

Part (b). Find

$$\frac{d}{d\theta}\int_{\pi/2}^{\pi-e^{-\theta}}\frac{\sin(x)}{x}\,dx.$$

Solution: To use the Second Fundamental Theorem of Calculus and the chain rule, we first do ourselves the favor of naming the integrand:

$$f(x) = \frac{\sin(x)}{x}.$$

Then the upper and lower limits are $u(\theta) = \pi - e^{-\theta}$ and $\ell(\theta) = \pi/2$, respectively. We drop the upper and lower limits into the integrand, factoring in the speeds of the limits:

$$\begin{aligned}\frac{d}{d\theta}\int_{\pi/2}^{\pi-e^{-\theta}}\frac{\sin(x)}{x}\,dx &= f(u(\theta))\cdot u'(\theta) - f(\ell(\theta))\cdot \ell'(\theta)\\ &= \frac{\sin(\pi-e^{-\theta})}{\pi-e^{-\theta}}\cdot e^{-\theta} - \frac{\sin(\pi/2)}{\pi/2}\cdot 0\\ &= \frac{\sin(\pi-e^{-\theta})}{\pi-e^{-\theta}}\cdot e^{-\theta}.\end{aligned}$$