Worksheet on optimization and related rates

MATH 124 · Calculus I · Section 26 · Fall 2008 Name ______________________

This worksheet is designed to walk you through some optimization and related-rate problems. On the first two pages, I give you some general guidelines for such problems. On subsequent pages, I give you some problems with substeps outlined for you. (On an exam, of course, you won't have such outlined substeps. This is practice.)

(Disclaimer: Some of these problems are adapted from Anton's *Calculus*, 3rd ed., Wiley, 1988.)

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Guidelines for optimization problems

(i) Draw a picture and label quantities.
(ii) Write down any expression(s) you have which relate the quantities.
(iii) Write down an expression for the quantity you want to optimize. Remember that *optimize* means (from its Latin root) *make best*. The word problem may say *minimize* or *maximize*; the techniques are the same in either case. (That's why we don't have separate textbook sections for minimization and maximization.)
(iv) Generally, that expression will depend on more than one variable. Use the relations from step (ii) to eliminate a variable.
(v) Differentiate the quantity to be optimized with respect to the remaining variable. Set that derivative to zero and solve.
(vi) Extrema occur at critical points (which you just found) or boundary points. Find the value at boundary points.
(vii) The max/min occurs the smallest/largest of the critical point(s) and the boundary points.
(viii) Sanity-check your answer.

Example: Find the dimensions of the rectangle of maximum area with perimeter 1000 feet.

(i) Draw a picture of a rectangle, labeling the length ℓ and width w.
(ii) The perimeter is $2\ell + 2w = 1000$ and the area is $A = \ell w$.
(iii) The quantity to be optimized is $A = \ell w$.
(iv) This depends on ℓ and w. Use the perimeter constraint to solve for ℓ: $\ell = 500 - w$. (You could also solve for w.) Then we have $A = \ell w = (500 - w)w = 500w - w^2$.
(v) The remaining variable is w. Differentiating gives us $dA/dw = 500 - 2w$. Setting this equal to zero we get $500 - 2w = 0$ so $w = 250$.
(vi) The smallest w can be is zero; this gives a rectangle of zero area. The largest w can be (due to the perimeter constraint) is 500. In that case ℓ is zero, also giving zero area.
(vii) So, the maximum area occurs when $w = 250$ (i.e. at the critical point, not at the boundary points).
(viii) Sanity check: Using the perimeter constraint we get $\ell = 250$ as well. The perimeter adds up to 1000; the length and width are all in a reasonable range (nothing is negative). (Can you think of anything else to check?)

Guidelines for related-rate problems

(i) Draw a picture.
(ii) Label quantities. Note which ones can vary, and which ones are held fixed.
(iii) Write down an expression relating the quantities. You do *not* have to solve for one variable in terms of the other. (You'll solve for the derivative below. This is kind of like implicit differentiation, where you solve for dy/dx even if you can't solve for y.) An equation relating the two quantities will suffice.
(iv) If there are more than 2 quantities which vary, use the additional information provided to eliminate variables until only two are left. (This step doesn't occur in every related-rate problem.)
(v) Take the derivative of both sides (almost always with respect to time). Make sure to use the chain rule whenever you encounter a quantity which varies with time. Remember that "rate of change" and "how fast" in the word problem both mean "take the derivative". You will get an equation with two derivatives in it.
(vi) You will have been given the value of one derivative. Solve for the unknown derivative.
(vii) Sanity-check your answer.

Example: An initially empty spherical balloon is inflated at a rate of 2 cubic inches per second. Find the rate of change of the radius of the balloon when the radius reaches 5 inches.

(i) Draw a picture of a sphere of radius r.
(ii) The volume V and the radius r both vary with time as the balloon inflates.
(iii) The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. There is no need to solve for r. We can solve for dr/dt later.
(iv) There are only two varying quantities (V and r), so there is nothing to eliminate.
(v) Differentiating both sides, we have $dV/dt = 4\pi r^2 dr/dt$.
(vi) We are given $r = 5$ and $dr/dt|_{t=5} = 2$. Plugging these in and solving for dr/dt we have

$$dr/dt = \frac{dV/dt}{4\pi r^2} = \frac{2}{100\pi} \approx 0.0064 \text{ in}^3/\text{sec}.$$

(vii) At radius $r = 5$, the volume is $4 \cdot 125\pi/3 \approx 524\text{in}^3$. Only two more cubic inches of air will be added in the next second. It makes sense that the radius would be changing by very little. Also, the radius is increasing, not decreasing, so apparently we didn't make a minus-sign error.

Problem 1. You are to make a box with square base and no top. You will (as guided below) find the dimensions that minimize the surface area of the box if the volume of the box is to be 1000 cm^3.

What do you call the three dimensions (front-to-back distance, left-to-right distance, bottom-to-top distance) of the box? The variable names are your choice.

What is the volume of a box in terms of those three lengths? Write an equation.

What does "square base" mean in terms of those three lengths?

Write an equation for $V = 1000$, in terms of those lengths. How many variables does V depend on?

What is the surface area A of the box? Remember we're not including the top. There are four sides, and the bottom. Those are all rectangles. What is the area of each? Remember that areas add. What is the sum of the 5 areas?

What is the name of the quantity we are asked to optimize?

How many variables does that quantity depend on?

Can those variables take on any value, or are they related somehow? Write down an equation involving them. Then solve that equation for one of the variables.

Plug that variable back into the area equation. Area should now depend on just one variable.

Differentiate that equation and set it to zero.

Solve for the remaining variable.

Now find the other dimensions of the box.

Does everything look reasonable? Why or why not?

Problem 2. You are to fence a rectangular area. The fencing for the left and right sides costs \$2/foot; the fencing for the front and back sides costs \$3/foot. Find the dimensions of the rectangular area that result in least cost, if the area is to be 3200 square feet.

I won't break down the steps for you on this one — do it on your own. Show each step.

(Hint to get you started: The cost of the fence is a concept somewhat related to the perimeter of the fence. If, for example, the depth is 10 feet and the width is 320 feet, then it would cost $2\cdot2\cdot10+3\cdot2\cdot320 = 1960$ dollars to make that fence.)

Problem 3. Grain pours from a chute at a rate of 8 cubic feet per minute. It makes a conical pile with the property that the height of the pile is always twice the radius. (See the Wikipedia article on *angle of repose* if you'd like to find out why this is a reasonable assumption.) How fast is the height changing when the height reaches 6 feet?

First I will tell you that the volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Which quantities in this equation are varying with time? Which are not?

I just wrote the equation for you in this case, so you don't need to. Note that we don't need to solve for h. Are there more than two time-varying quantities? If so, how can you eliminate one of them? Remember that the question is asking about rate of change of height, so the height variable needs to survive this elimination process.

Differentiate both sides of the equation with respect to time.

How many derivatives do you see? Which one's value did I give you? Which one's value did I ask you to solve for?

Solve for the requested derivative.

Does your answer make sense? Why?

Problem 4. An aircraft climbs at an angle of 30° to the horizontal. How fast is the altitude changing if the air speed is 500 miles per hour?

I won't break down the steps for you on this one — do it on your own. Show each step.

(Hint: Draw a triangle with its base on the ground. The altitude of the plane is straight up. Which direction is the plane flying in? It rhymes with *flypotenuse*.)