PROFESSIONAL DEVELOPMENT WORKSHOP · SPRING 2006
ASSIGNMENT 11 · A LESSON PLAN ON WORD PROBLEMS

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I talk about word problems quite a bit throughout the semester. What follows contains some snippets from the first day of the course, as well as from a sample day in the course.

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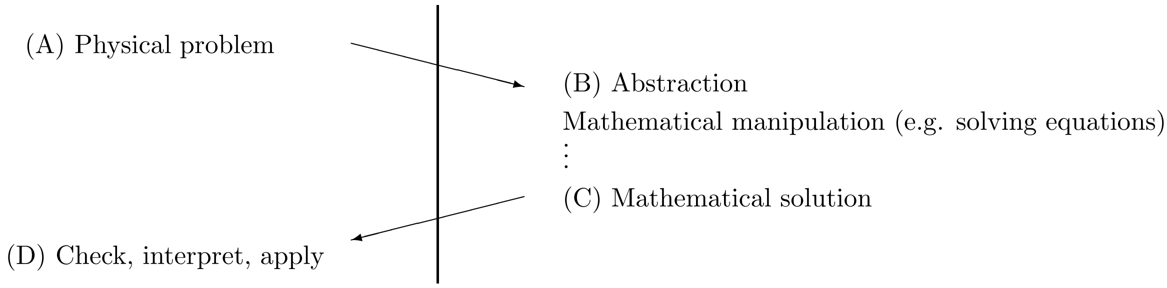
The day-one part: Where math comes from is as follows: we are presented with a physical problem — surveying, finances, construction, etc. We then abstract the problem and learn to solve it mathematically. Then, we apply our results back to the physical situation:



Now, there are people in the world, called "pure mathematicians," who work on abstract problems for their own sake, and don't worry too much about reconnecting with applied problems. But for this course, we are interested in math for its applications — whether to things you'll encounter later, in your working careers, or sooner, say in your lab-science classes.

I know that the phrase "word problem" has a bad connotation for many students, and poorly written word problems from our high-school days are in part to blame for that. But I also think another reason for the bad reputation is that they're *hard*. Most students, once they get the math set up (i.e. after they get from point A to point B in the above diagram), are able to proceed (modulo arithmetic and algebra errors). The hard problem is getting the problem set up, and in fact the most frequent complaint is *I don't know where to start*. There is nothing to be surprised or ashamed about — this is a common state of human existence, so we should get used to it.

I think of word-problem skills as the math-course version of problem-solving skills. In this course we will spend a lot of time talking about *what to do when you don't know what to do.* I don't kid myself that most of you will need to find the domain of a rational function ever again in your lives after college algebra. But in your careers, what will happen is that you and your co-workers will be standing around something broken — a piece of equipment, a spreadsheet that won't add up, disappointing sales figures, etc. — and no one will know what to do, but you will need find out how to do something anyway.

Getting from point A to point B can be hard; from point B to point C is often easier. Often we forget to go from point C to point D altogether. So the second key piece of problem solving is to ask: *Are my answers making sense?* I phrase this in terms of: don't believe whatever comes out of your calculator. Don't believe

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whatever a high-priced consultant tells you. And most importantly don't believe whatever a slick salesperson tells you just because you figure he knows his business.

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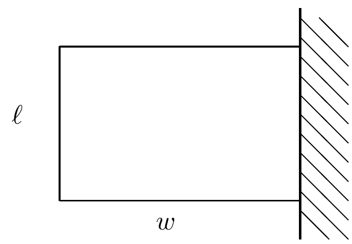
Sample problem: A rectangular fence is to be built against a wall (so the fence will have three sides), using a hundred feet of fencing. Find the dimensions of the fence that will give the maximum area.

What's the answer here? *I have no idea!* The point is that for simple problems, in fact many of the problems you've had in your academic careers up to this point, you could see what to do and you could go ahead and do it. But don't always expect that — for harder problems, in fact for pretty much any problem worth doing, we won't know what to do.

Problems differ; there is no single recipe. However, some guidelines apply. Here are some things I always want you to do:

- Draw a picture whenever possible.
- Label the quantities. Include the units of measurement — these contain valuable information and can help you set up the problem, and/or catch some mistakes. For example, in rate problems, we write $D = RT$. With units only, that's, say, miles = (miles/hour) times hours. The dimensions work out to be miles on both sides of the equation, which is reassuring.
- Write down how those quantities are related.
- If you have numbers, use them. Else leave them variables for the time being.
- Whenever you have equations with more than one variable, eliminiate one variable at at time. It doesn't have to be in the right order. You'll get there. Key point: we are good at solving single-variable equations. Then, once you've solved for that variable, back-substitute to find the values of the remaining variable(s).
- Interpret the answers. What does the answer mean? What are the units of measurement?
- Sanity-check. Is the wind blowing backward, or at 20,000 miles per hour? If you deposited $400 at 3 percent for two years and you come up with 15 trillion dollars, something is wrong. Go back and find it.

Here, what kind of picture can we draw? We can draw the wall, and the rectangle shape up against it. We don't know the dimensions yet — how long and wide the rectangle will be — so we'll have to leave them as variables for now:



One of the things that's hard about the problem as stated is that *there are no numbers in it.* It doesn't even look like a math problem at all. OK, so how do we turn it into a math problem we can solve? We don't always know. One of my problem-solving guidelines is: Write down what you *do* know and see if the view improves. What is the "maximum area"? I have no idea either! But we can write down what the area is, and put the maximum part on the back burner for now. That's $A = \ell w$. This is already looking pretty good — it's beginning to look like a math problem.

What else do we know? What's the "hundred feet of fencing" business about? OK, so however we lay the fence, it will be in a rectangle shape rather than say a triangle or a semicircle or whatever. It will go from the wall, out to the left edge, down, and back — so how can we write that? OK, w, plus ℓ, plus w, adds up to 100 — that's $2w + \ell = 100$.

OK, so now we have two equations:

$$\begin{aligned} A &= \ell w \\ 2w + \ell &= 100. \end{aligned}$$

Now what? Well, remember that we're really pretty good at solving certain kinds of equations in one variable — linear, quadratic, etc. But here, we have one variable too many, so we should eliminate one. How do we actually *do* that? When you you think *eliminate*, you should think *solve for.* So let's pick a variable — let's not worry about whether we've got the "right" one. What have we got? Well, let's write

$$\ell = 100 - 2w.$$

Now that we have another way to write ℓ, let's put it into the area equation, since we are trying to "maximize area" in some way or other. Again, I don't know a priori how to find the maximum of this or that, but let's go ahead and write it down and see if the view improves. What do we get?

$$A = \ell w = w(100 - 2w) = 100w - 2w^2.$$

Now what? Whenever you're looking at an equation, ask yourself what *kind* of equation it is — is the variable appearing in a denominator? Is it a polynomial equation? If so, what's the highest power of the variable appearing? OK, the highest power of w is 2. What kind of equation is this? Right, it is a *quadratic equation.* (At the moment, that's a good sign since we're in section 2.2 talking about quadratic equations. On the exam, or the final, or when you're using math later on, though, you'll need to find out what kind of equation you are looking at.)

What do we know about the maximum of a quadratic equation? Right, the graph of a quadratic equation is a parabola, and the maximum (or minimum) appears at the *vertex*, so we could find that. How? OK, good, we can *graph* this function, which would be a graphical technique, or we could use the formula for the vertex of a parabola. (Or, both — which would make a nice check.) Where's the x, though? OK, so w is the variable . . . let's write down the equation in the usual order:

$$A = -2w^2 + 100w.$$

So the a is -2, the b is 100, and the c is — not there, so 0. We have

$$h = \frac{-b}{2a} = \frac{-100}{-4} = 25.$$

So what does this mean? Does it mean the answer to the original maximum-area question is 25? Remember that when we write (h, k) coordinates of the vertex, the h is the *input* coordinate of where the maximum is, and the k is the *output* coordinate of where the maximum is. What are the input and output to the function? OK, you put in a w and you get out an A. So 25 is the width giving the maximum area. What *is* the maximum area? Well,

$$A(25) = 100 \cdot 25 - 2 \cdot 25^2 = 2500 - 2 \cdot 625 = 2500 - 1250 = 1250.$$

Are we done yet? We know the maximum area we can enclose, and we know how wide to make the fence. The other dimension we still need to find is the length. How are those two related? Right, we know that

$$2w + \ell = 100.$$

So now that we know what number w is,

$$\ell = 100 - 2w = 100 - 2 \cdot 25 = 100 - 50 = 50.$$

Great — so now we know that if we make the fence 25 feet wide and 50 feet long, we'll have enclosed an area of 1250 square feet, which is the best we can do with 100 feet of fence.

Should we move on to the next problem? I don't know about you but I never get a good solid feeling about my answer until I've checked it out. What can we do here? Is everything reasonable? Well, 25+50+25 is indeed 100, so that checks out. Also nothing came out negative, so — it looks like we're good to go.