Exam #2 solutions · Thursday, March 1, 2007

MATH 124 · Calculus I · Section 8 · Spring 2007

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, except for parts 2c, 2d, and 8b which specifically ask for verbal responses.
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Problem 1. Let $f(x) = x^x$. Numerically approximate $f'(2)$ using difference quotients. Use at least three successively smaller values of h.
Solution: The approximation is $f'(x) \approx \frac{f(x+h)-f(x)}{h}$. We have $x = 2$ throughout, $f(x) = x^x$, and $f(x+h) = (x+h)^{x+h}$. With $h = 0.01$:

$$\frac{f(x+h) - f(x)}{h} = \frac{2.01^{2.01} - 2^2}{0.01} \approx 6.840$$

With $h = 0.001$:

$$\frac{f(x+h) - f(x)}{h} = \frac{2.001^{2.001} - 2^2}{0.001} \approx 6.779$$

With $h = 0.0001$:

$$\frac{f(x+h) - f(x)}{h} = \frac{2.0001^{2.0001} - 2^2}{0.0001} \approx 6.773.$$

It looks like about 6.77.

Problem 2. On a mountain-climbing expedition, you find that the air becomes cooler as you climb. Let y be your altitude above sea level, measured in feet; let H be the air temperature in degrees Fahrenheit.

Part (a). What are the units of $H'(y)$?
Solution: The notation tells you that H is a function of y. The input units of $H(y)$ are the units of y, which are feet. The output units of $H(y)$ are the units of H, which are degrees Fahrenheit. The units of the derivative $H'(y)$ are the original function's output units over the original functions input units, namely, degrees Fahrenheit per foot.

Part (b). What is the sign of $H'(y)$?
Solution: The air temperature is getting cooler (decreasing number of degrees) as you climb (increasing number of feet). So, the sign must be negative.

Part (c). Give a practical interpretation of $H^{-1}(35)$.
Solution: A function sends inputs (here, feet) to outputs (here, degrees Fahrenheit); the inverse function sends the original function's outputs (degrees) back to the original function's inputs (feet). So, 35 must be degrees Fahrenheit. Then $H^{-1}(35)$ must be the altitude at which the temperature is $35°F$.

Part (d). Give a practical interpretation of $H'(7500)$.
Solution: The derivative $H'(y)$ is the rate of change in temperature as a function of altitude. At altitude 7500 feet, the temperature is dropping by this many degrees per additional foot of altitude.

Problem 3. In a lab experiment, you have microorganisms growing in a Petri dish. The number m of microrganisms, in millions, is a function of time t in days since the start of the experiment. This number is given by

$$m(t) = 4.1e^{0.24t}.$$

Find the rate of change in population on day 5 of the experiment. In your answer, please show units.

Solution: The derivative $m'(t)$ is $0.24 \cdot 4.1e^{0.24t}$, using the chain and exponential rules. Evaluating at $t = 5$ gives

$$m'(5) = 0.24 \cdot 4.1e^{0.24 \cdot 5} = 0.984e^{1.2} \approx 3.267 \text{ million microorganisms per day.}$$

Problem 4. Let $G(t) = 2^{-rt}\sin(at)$. Find $G'(t)$.
Solution: Using the product and chain rules, we have

$$G'(t) = -r\ln(2)2^{-rt}\sin(at) + a2^{-rt}\cos(at).$$

Problem 5. Let $f(z) = \tan^{-1}(z)$. Compute $f''(1)$.
Solution: The first derivative is

$$f'(z) = \frac{1}{1+z^2}.$$

For the second derivative, you can use the quotient rule, or you can use the exponential and chain rules since we can write $f'(z) = (1+z^2)^{-1}$. Either way, you should get

$$f''(z) = \frac{-2z}{(1+z^2)^2}.$$

Evaluating at $z = 1$ gives

$$f''(1) = \frac{-2}{(1+1^2)^2} = \frac{-1}{2}.$$

Problem 6. Let $q(x) = \ln(2 + 2x + x^2)$.

Part (a). Find $q'(x)$.
Solution: Using the log and chain rules, we have

$$q'(x) = \frac{2+2x}{2+2x+x^2}.$$

Part (b). Find an equation for the tangent line to $q(x)$ at $x = 3$.
Solution: As usual, use point-slope form at the specified point. The horizontal coordinate of the point is $x_0 = 3$; the vertical coordinate of the point is

$$y_0 = q(3) = \ln(2 + 6 + 9) = \ln(17).$$

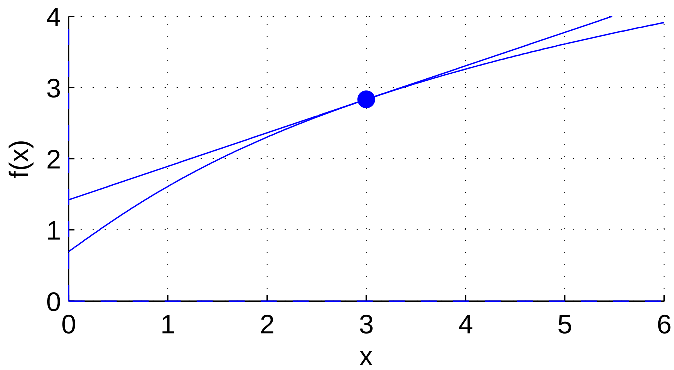
The slope is

$$m = q'(3) = \frac{2+6}{2+6+9} = \frac{8}{17}.$$

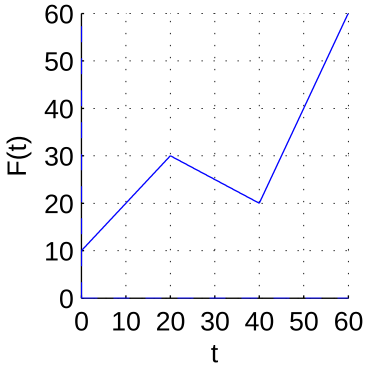
Putting these together we have

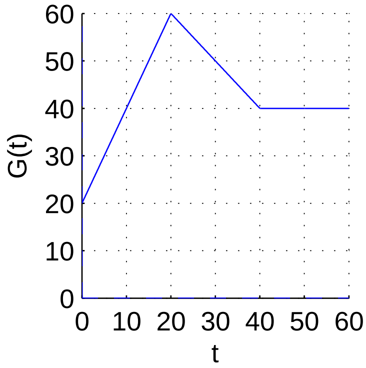
$$\begin{aligned} y &= y_0 + m(x - x_0) \\ y &= \ln(17) + \frac{8}{17}(x-3). \end{aligned}$$

As a sanity check, you can graph the original function and the tangent line. You should see



Problem 7. Let $F(t)$ and $G(t)$ be given by the following graphs.





Part (a). Find $G'(40)$.
Solution: This is undefined: G has a corner at $t = 40$.

Part (b). Let $H(t) = \frac{F(t)}{G(t)}$. Find $H'(50)$.
Solution: Using the quotient rule, we have

$$H'(t) = \frac{F'(t)G(t) - F(t)G'(t)}{G(t)^2}.$$

At $t = 50$, this is

$$H'(50) = \frac{F'(50)G(50) - F(50)G'(50)}{G(50)^2}.$$

We can read all these values off the graph:

$$F(50) = 40, \qquad F'(50) = 2, \qquad G(50) = 40, \qquad G'(50) = 0.$$

Then

$$H'(50) = \frac{2 \cdot 40 - 40 \cdot 0}{40^2} = \frac{80}{1600} = 0.05.$$

Part (c). Let $H(t) = F(G(t))$. Find $H'(30)$.
Solution: Using the chain rule, we have

$$H'(t) = F'(G(t))G'(t).$$

We can read all the necessary values off the graph:

$$G(30) = 50, \qquad F'(50) = 2, \qquad G'(30) = -1.$$

Then

$$H'(30) = F'(G(30))G'(30) = 2 \cdot (-1) = -2.$$

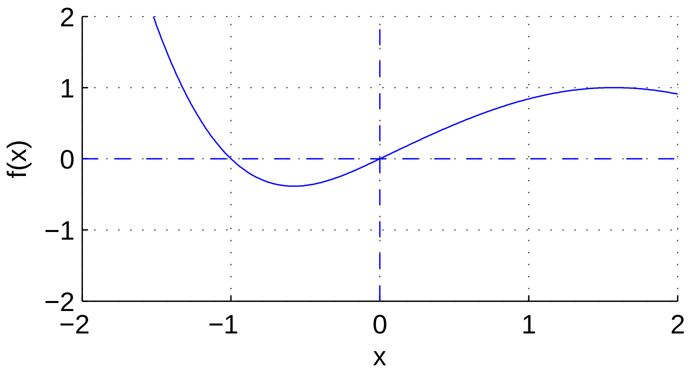
Problem 8. Let

$$f(x) = \begin{cases} \sin(x), & x \geq 0 \\ x - x^3, & x \leq 0. \end{cases}$$

Part (a). Find $f'(x)$. Write it as a piecewise function.
Solution: Differentiating the pieces, we have

$$f'(x) = \begin{cases} \cos(x), & x \geq 0 \\ 1 - 3x^2, & x \leq 0. \end{cases}$$

Part (b). Is the original function $f(x)$ differentiable at $x = 0$? Why or why not? (Hint: graph it.)
Solution: Graphing the function, you should see



The slopes match up at $x = 0$ so $f'(x)$ is defined there.

Problem 9. When is $g(x) = x^3 + bx^2 + cx + d$ concave up? Assume b, c, d are constants. (You will need to solve an inequality.)
Solution: A function is concave up when $g''(x) > 0$. Computing this, we have

$$\begin{aligned} g(x) &= x^3 + bx^2 + cx + d \\ g'(x) &= 3x^2 + 2bx + c \\ g''(x) &= 6x + 2b \\ g''(x) &> 0 \\ 6x + 2b &> 0 \\ 6x &> -2b \\ x &> \frac{-b}{3}. \end{aligned}$$