Exam #3 study guide · Math 124 · Calculus I · Section 8 · Spring 2007

Disclaimers about the study guide:

- Exam 3 covers sections 3.7 through 4.7. (Remember we skipped sections 3.10 and 4.4). While all *topics* on the exam will be taken from this study guide, the specific *questions* on the exam will not be identical to the ones you see here.
- In addition to consulting this guide, please review all homework problems for sections 3.7 through 4.7. In particular, look at unassigned problems nearby. For example, if I assigned #14, see if you can do #13 and #15.
- For reference, you can: (*) use the back of the book; (*) use the student solution manual; (*) make use of the tutor center in Math East 145; (*) ask questions in class; (*) talk to me after class, or during office hours.

John Kerl
`kerl at math dot arizona dot edu`
2007-03-26

* * *

Topics:

- Even though sections 3.1 through 3.6 were on the last exam, we're still using everything from chapter 3. You very much still need to know the product rule, quotient rule, chain rule, derivatives of sin, cos, tan, $\sin^{-1}$, $\tan^{-1}$, ln, etc.
- Implicit differentiation: Given an equation relating two variables (say, y and x), differentiate both sides and solve for dy/dx. Remember to use the chain rule since y depends on x.
  Example:

$$\begin{aligned} x^3 + y^2 &= 3 \\ 3x^2 + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-3x^2}{2y}. \end{aligned}$$

- Hyperbolic functions:
  - Definitions:

$$\begin{aligned} \sinh(x) &= \frac{1}{2}(e^x - e^{-x}), \\ \cosh(x) &= \frac{1}{2}(e^x + e^{-x}), \\ \tanh(x) &= \sinh(x)/\cosh(x). \end{aligned}$$

  - Be able to differentiate all three (use definitions and the chain rule, or memorize):

$$\begin{aligned} d/dx(\sinh(x)) &= \cosh(x), \\ d/dx(\cosh(x)) &= \sinh(x), \\ d/dx(\tanh(x)) &= 1/\cosh^2(x). \end{aligned}$$

  - My primary interest in these functions for calculus per se is to add some variety to chain-rule problems. Also, having you prove hyperbolic identities such as $\cosh^2(x) - \sinh^2(x) = 1$ is an algebra exercise. (It checks your proficiency with exponent rules.)
- Linear approximation:
  - Know how to write an equation for the tangent line $\ell(x)$ to a function $f(x)$. Remember: point-slope form $y = y_0 + m(x - x_0)$ with $y_0 = f(x_0)$ and $m = f'(x_0)$.

  - Graph and check for tangency at x_0.
  - Be able to write down the error function $E(x) = f(x) - \ell(x)$; graph it on an interval and see what the worst-case error is; see if the linear approximation is an overestimate or an underestimate.
  - Remember that if $f''(x_0) > 0$ (concave up there), the function will curve up away from the tangent line; if $f''(x_0) < 0$ (concave down there), the function will curve down away from the tangent line.
  - Example: e^x near $x_0 = 1$. $f(1) = e$; and $f'(x) = e^x$ so $f'(1) = e$. Then $\ell(x) = e+e(x-1) = ex$. Graph both e^x and ex and observe that the tangent line is below the original function so the linear approximation is an underapproximation. We expect this since e^x is concave up everywhere, so certainly concave up at 1. Error function is $E(x) = e^x - ex$. Graph this function on $[0, 2]$ and tell me for what x it's biggest ($x = 2$).
- Using first and second derivatives:
  - Critical points: derivative is zero or undefined.
  - Local maxima and minima: occur at critical points or boundary points.
  - How to tell if a critical point is a local maximum, local minimum, or a false alarm: graph, first derivative test, second derivative test. Graphing usually suffices unless there are parameters involved.
  - Inflection points: where the graph changes concavity. Be able to compute a second derivative and solve for x. Use the graph to determine if the point you find is an inflection point or a false alarm. (Examples: x^4 and x^3 at $x = 0$. Both have zero second derivatives there; only the latter function actually changes concavity there.)
- Families of curves:
  - Be able to experimentally find out what effect a parameter has. E.g. for $f(x) = e^{-ax}$, plot it for bigger and smaller values of a and tell me what they do. (The more you remember from transformations of functions in college algebra / precalculus, the easier this will be.)
  - If the first derivative is positive/negative at a point, the original function is increasing/decreasing there, respectively.
  - If the second derivative is positive/negative at a point, the original function is concave up/down there, respectively.
  - Find parameters to make a function satisfy certain constraints, including horizontal/vertical asymptotes, x- and y-intercepts, critical points, inflection points.
    Example: Let $f(x) = x^2 + bx + c$. Find b and c such that $f(x)$ has a y-intercept of 2 and a critical point at $x = 3/2$. The y-intercept is $f(0) = c = 2$. Critical point at $x = 3/2$ means $f'(3/2) = 0$. Compute $f'(x) = 2x + b$; then $f'(3/2) = 3 + b = 0$ so $b = -3$.
    Example: Find a so that $f(x) = x^3+ax^2+bx+c$ has an inflection point at $x = 4$. Compute $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. Then we want $f''(4) = 0$ so $24 + 2a = 0$ so $a = -12$. Then b and c can be anything, say 0. Graph $x^3 - 12x^2$ to check. Does the concavity change at $x = 4$?
- Optimization:
  - Global maxima/minima of continuous functions occur only at critical points and boundary points.
  - Upper/lower bounds are either global max/min (function reaches its extreme output values) or asymptotes (function approaches its extreme output values).
- Optimization and modeling:
  - Draw a picture. Label quantities and their relationships (these will become equations).
  - Know what you want to optimize.
  - Know what you can vary.
  - Often, you will have an equation with more than one input. Eliminate a variable until the quantity you want to optimize is a function of one variable.
  - Solve the 4.3 problem.

  - Example: Rectangle of maximum area with perimeter 200. Want to optimize $A = xy$. This has two inputs, x and y. Since $2x + 2y = 200$, eliminate $y = 100 - x$. Maximize $A = x(100 - x)$.
  - You will often need to know some geometry/trig facts to set these problems up, such as:
    * Definitions of sine, cosine, and tangent in terms of sides of a triangle (SOHCAH-TOA).
    * Pythagorean theorem.
    * Equation for a circle of radius r.
    * Area of a circle, rectangle, triangle.
    * Volume of a rectangular box or circular cylinder.
  - Also remember that areas of complicated objects (e.g. surface area of box with no lid — I didn't memorize that formula!?!) are built up of simpler pieces (areas of the four sides and the bottom, all five of which are rectangles).
- Related rates:
  - You have two quantities which are related by an equation.
  - They both vary with respect to some variable (often, time).
  - You know the rate of change of one quantity, either as a formula or for a single data point.
  - Differentiate both sides (use the chain rule) and solve for the unknown rate of change.
  - As with optimization problems, you may need to eliminate a variable.
  - Example: $A = \pi r^2$ and $dr/dt = 0.4$. Find dA/dt at $r = 6$. Differentiating both sides and using the chain rule: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Then $\frac{dA}{dt}|_{r=6} = 2\pi \cdot 6 \cdot 0.4 = 4.8\pi$.
- L'Hôpital's rule:
  - If $f(x)$ and $g(x)$ are differentiable at a point a, $f(a) = g(a) = 0$, and $g'(a) = 0$, then $\lim_{x\to a} f(x)/g(x) = f'(a)/g'(a)$.
  - Example: $h(x) = \sin(x)/x$. Graph it and see that it looks like $h(0)$ should be 1. Using l'Hôpital's rule, $\lim_{x\to 0} h(x) = \cos(0)/1 = 1$ as expected.
  - Also works for limits at infinity (horizontal asymptotes). In particular, this gives us yet another way to know that the horizontal asymptote of a balanced rational function is the ratio of leading coefficients (use l'Hôpital's rule repeatedly).
  - Use it in the following situations:

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{0}{0} \\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{\infty}{\infty} \\
\lim_{x\to \pm\infty} \frac{f(x)}{g(x)} &= \frac{0}{0} \\
\lim_{x\to \pm\infty} \frac{f(x)}{g(x)} &= \frac{\infty}{\infty}
\end{aligned}$$

  - Know when this rule applies and when it does not. Example:

$$h(x) = f(x)/g(x) = (x + 1)/(x + 2).$$

    Then $f'(x)/g'(x) = 1/1 = 1$ everywhere, but $h(3) \neq 1$. What went wrong?