John Kerl
Math. 473
University of Arizona
Dec. 1, 1993

Some pretty patterns on the Rubik's Cube

The Rubik's Cube, also known as the Hungarian Magic Cube, has long been a laboratory for mathematicians interested in group theory. While the great number of patterns possible on the cube is legendary, there is a much smaller (and personally defined) subset of patterns that are aesthetically pleasing. Many of these patterns, sometimes known as "pretty patterns", are pleasing because they embody some symmetries. This paper will present a particular group of patterns that embody a certain symmetry and will provide techniques for producing those patterns.

The patterns of the cube are *permutations* of the pieces and orientations. A permutation on a set is a function defined from a set to itself such that every element has exactly one image. Thus, permutations are defined on every member of the set, and can always be "undone" — they are *invertible*. A permutation may be thought of as a rearranging — for example, shuffling a deck of cards is a permutation on the deck.

It was stated above that many of the "pretty" permutations of the cube may be thought of as embodying certain symmetries. To make this vague statement more clear, consider a square with the four vertices labeled A, B, C and D (fig. 1). There are eight transformations, known as the *symmetries of the square*, that carry the square back onto itself: rotations through 0° (the *identity* transformation), 90°, 180° and 270°; and reflections through the vertical, horizontal, and either of the two diagonal axes (fig. 2). The labels of the corners shown in fig. 2 represent the square following a rotation right by 90°, or by a reflection through the vertical axis followed by a reflection through the axis running from upper left to lower right.

There are, of course, many more than eight *symmetries of the cube*, by which I mean the plain, geometric cube (fig. 3); one may have 24 reflections and 24 rotations, rather than the four of each in the case of the square. The symmetry that I will discuss is the rotation through an axis running through opposite corners. (It is in this sense that it was stated above that a particular group of patterns embodying a certain symmetry was to be considered.) Consider fig. 3. A line runs through the corners labeled B and E. The corners might be rotated as follows: A goes to G, G goes to C, C goes to A; H goes to F, F goes to D, D goes to H. Now consider the Rubik's Cube (fig. 4). It would be interesting to be able to *cycle* the pieces labeled with the same numbers on the Rubik's Cube in a way similar to the way in which the corners A, G, C or H, F, D of the plain cube were rotated: for example, to cycle all edge pieces labeled with a 5, just as the corners A, G, C can be rotated. In the case of the corner piece, we would simply like to rotate it (and its far-corner opposite) in place. In fact, all these patterns are possible. I will not justify this claim, merely noting instead that, whereas not all permutations of the pieces and their orientations can be achieved by manipulating the cube (without disassembling and reassembling it), the ones just described are possible. (The interested and/or mathematically inclined reader is advised that only *even* permutations of positions of edges or corners are possible, and that flips of edges or twists or corners are possible provided that the sum of the spins is zero, modulo 2 or 3 respectively.)

The pretty patterns to be considered are clockwise cycles of the pieces noted in fig. 4. Before presenting the actual *processes* (sequences of moves) to achieve these patterns, a little notation must be devised. Pick a face of the cube; consider this to be the top face. Of the remaining four side faces, pick one to be the front. In this orientation, the faces may be labeled U, D, F, B, L and R, for up, down, front, back, left and right (fig. 5). Denote a clockwise (as viewed from outside the cube) quarter-turn of a given face by its letter. Then a half turn of the face will be denoted by the exponent 2, and a counterclockwise quarter turn will be denoted by the exponent -1. A process will be denoted by the string of letters and their associated exponents, e.g. UF^2R^{-1} means turn the upper face a quarter-turn clockwise, then turn the front face a half turn, then turn the right face a quarter-turn counterclockwise. Spaces or dots have no significance other than to separate mnemonic units. Processes are given in the following table.

Pieces to be cycled (fig. 4)	*String*
1	$B^2 \cdot R^{-1}DR\,FDF^{-1} \cdot U \cdot FD^{-1}F^{-1}\,R^{-1}D^{-1}R \cdot U^{-1} \cdot B^2$
2 (do parts a and b)	$(a)\ R \cdot U^{-1}\,R^{-1}F^{-1}L^{-1}B^{-1} \cdot U^{-1}BLFR \cdot U^2 \cdot R^{-1}$
	$(b)\ L^{-1} \cdot D\,LBRF \cdot D\,F^{-1}R^{-1}B^{-1}L^{-1} \cdot D^2 \cdot L$
3	$F^2U^2F^{-1}R^2F \cdot 2a \cdot F^{-1}R^2FU^2F^2$
4	$FL^2F^{-1} \cdot LFR^{-1}F^{-1}L^{-1}FRF^{-1} \cdot FL^2F^{-1}$
5	$F^2R^2FU^2F^{-1} \cdot 2a \cdot FU^2F^{-1}R^2F^2$
6	$R^{-1}F \cdot L^{-1}URU^{-1}LUR^{-1}U^{-1} \cdot F^{-1}R$

It is clear that any of the above processes may be done once, twice, or no times, and that each may be done independently of the rest. Thus there are $3^6 = 729$ different patterns in this small group alone, a very small corner of the group of the more than 4.3×10^{19} permutations of the Rubik's Cube.

```
                     a         a         another
                     diagonal  vertical  diagonal
                     axis      axis      axis          H +-----------+ G
A +-----------+ B    D +-----|-----+ A                 / |         / |
  |           |        | \   |   / |                A +-----------+ B|
  |           |        |   \ | /   |                  |  |        |  |
  |           |       -------X------- horizontal      |  |        |  |
  |           |        |   / | \   |  axis            |  |        |  |
  |           |        | /   |   \ |                  | E+--------|--+ F
D +-----------+ C    C +-----|-----+ B                | /         | /
      Fig. 1               Fig. 2                   D +-----------+ C
                                                          Fig. 3
```

```
       +-----+-----+-----+                 +-----+-----+-----+
      /  6  /  3  /     4|                /     /     /     /|
     +-----+-----+-----+ |               +-----+-----+-----+ |
    /  5  /     /     2| |              /     /  U  /     /| |
   +-----+-----+-----+ |/|             +-----+-----+-----+ |/|
  /     /     /     /| / |            /     /     /     /| / | <--- B
 +--4--+--2--+-----1 |/|5|           +-----+-----+-----+ |/| |
 |     |     |     | / |/|           |     |     |     | /R|/|
 |     |     |     |/| / |     L--> |     |     |     |/| / |
 +-----+-----+-----+ |/|6/           +-----+-----+-----+ |/| /
 |  3  |     |     2 / |/            |     |  F  |     | / |/
 |     |     |     |/|3/             |     |     |     |/| /
 +-----+-----+-----+ |/              +-----+-----+-----+ |/ <--- D
 |  6  |  5  |     4 /               |     |     |     | /
 |     |     |     |/                |     |     |     |/
 +-----+-----+-----+                 +-----+-----+-----+
       Fig. 4                           Fig. 5
```