DISTRIBUTIONAL PROPERTIES OF BRIDGE DECOMPOSITIONS OF UPSAW

Abstract.

1. Introduction

The study of self-avoiding random walks goes back at least to the 1950's. Even in the special case of random walks supported on regular 2-d lattices, there are several types of self-avoiding random walks that have been intensely researched by mathematicians and physicists. These include but are not limited to the self-avoiding walk, the myopic self-avoiding walk, the loop-erased random walk and the percolation exploration process. Here we are specifically interested in processes supported on subsets of the square lattice.

Kesten's 1963 article, "On the number of SAWs" garners interest in the subject by introducing a simply posed question, 'how many SAWs of length n on a square lattice are there?', that still cannot be solved rigorously and is computationally unfeasible even for relatively small n. Moreover, many details of the asymptotic behavior of the number of SAWs of length n as $n \to \infty$ still alludes derivation.

Precisely, an *n-step self-avoiding walk* is a sequence of lattice sites $[\omega(0), \omega(1), \ldots, \omega(n)] \subset \mathbb{Z} + i\mathbb{Z}$ satisfying

- $\omega(0) = 0$,
- $|\omega(j+1) - \omega(j)| = 1$
- $\omega(i) \neq \omega(j)$ for $i \neq j$

We use the notation C_n to denote the number of SAWs of length n. Kesten recognized that C_n quickly becomes uncomputable with the size of n and conjectured that

$$c_N \sim A\beta_c^{-N} N^{\gamma - 1},$$

for some constants $\beta_c, A, \gamma > 0$. It can be shown that

$$\beta_c := (\lim_{N\to\infty} c_N^{1/N})^{-1},$$

exists and is finite. The parameter $1/\beta_c$ is called the *connective constant* of SAW.

We have not yet defined a probability measure on the n-step SAWs. The *n-step self-avoiding random walk* is defined to be the uniform measure on n-step SAWs. The *whole-plane self-avoiding random walk (starting at the origin)* or *infinite whole plane SAW (starting at the origin)* is defined to be the limit as $n \to \infty$ of the uniform measure on n-step SAWs.

One specific variation of the infinite self-avoiding random walk we consider here is the infinite half-plane SAW. We define this random walk by first restricting our n-step SAWs to those with $\mathrm{Im}[\omega_j] > 0$ for all $j > 0$; such a walk is called an *n-step (upper) half-plane SAW (starting at the origin)*. The *infinite half-plane SAW* (UPSAW) is defined to be the limit as $n \to \infty$ of the uniform measure on n-step half-plane SAWs. This is not the only way

UPSAW can be defined; existence of the limit and alternate methods of defining the measure are reviewed Section 2.

The infinite half-plane SAW is supported on $\mathbb{Z} + i\mathbb{N}$, with $\mathbb{N} := \{0, 1, 2, \dots\}$. In the context of the present study, it is intuitively helpful to allow the mesh width of our square half-plane lattice to vary; that is, we consider the same probability measure defined above on the lattice $\delta\mathbb{Z} + i\delta\mathbb{N}$ for some *mesh width* $\delta > 0$. It is thought that these probability measures converge to a non-degenerate probability measure on simple paths supported on $\mathbb{H} := \{z \in \mathbb{C} : \text{Im}[z] \geq 0\}$ as $\delta \to 0$. Recent advances in mathematical physics support this conjecture and provide a likely candidate for the scaling limit [7]. The candidate is denoted chordal $\text{SLE}_{8/3}$; some details on the origins and properties of chordal $\text{SLE}_{8/3}$ are provided in Section 3.

Another variation of the infinite self-avoiding random walk and the primary object of this study is self-avoiding random walk in the *k-strip* defined by $\mathbb{Z} + i\{0, 1, 2, \dots k\}$ with $k < \infty$. Fixing $k \in \mathbb{N}/\{0\}$, self-avoiding walks $\omega := [\omega(0), \omega(1), \dots, \omega(n)] \subset \mathbb{Z} + i\{0, 1, \dots k\}$ that we consider here satisfy

- $n \geq k$,
- $\omega(0) = 0$, $\text{Im}[\omega(n)] = k$
- $|\omega(j+1) - \omega(j)| = 1$, and
- $\omega(i) \neq \omega(j)$ for $i \neq j$.

Use $|\omega|$ to denote the length, n, of such a walk. We define the *self-avoiding random walk on the k-strip* to be the probability measure on such walks given by,

$$\mathbf{P}[\omega] := \frac{\beta_c^{-|\omega|}}{Z} \text{ where } Z := \sum_{|\omega| \geq k} \beta_c^{-|\omega|}$$

Note that this is equivalent to defining a probability measure on the lattice strip given by $\delta\mathbb{Z} + i\{0, \delta, 2\delta \dots k\delta\}$ with $\delta = 1/k$ or the *normalized k-strip*. Motivations similar to those that led to the conjecture that the fine-mesh scaling limit of UPSAW converges to chordal $\text{SLE}_{8/3}$ also lead to a conjecture that self-avoiding random walk on the normalized k-strip conditioned to terminate at a point $x + i \in \mathbb{H}$, with $x \in \mathbb{R}$ fixed, converges to chordal $\text{SLE}_{8/3}$ as $k \to \infty$ (due to a property of $\text{SLE}_{8/3}$ called conformal invariance, chordal $\text{SLE}_{8/3}$ is well-defined in any simply connected domain with specified initial and terminal points).

In this paper we use the conjecture that UPSAW and SAW on the normalized k-strip conditioned to terminate at $x + i$ converge to chordal $\text{SLE}_{8/3}$ in the upper half plane and chordal $SLE_{8/3}$ on the strip (initialized at 0 and terminating at $x+i$), respectively, to predict certain random variables associated with SAW on the normalized k-strip for large k, say $k >> 0$. We then conduct numerical experiments to test the accuracy of these predictions.

In Section 2.1, we take a slight detour that reviews an alternate definition of the UPSAW in terms of infinite concatenations of objects called irreducible bridges. Bridges are special cases of self-avoiding walks, ω, defined precisely in Subsection 2.1; these objects also live in k-strips or, alternatively, normalized k strips, depending on the context. Theorems in Section 2.1 will allow us to conclude that for fixed $k > 0$, distributions of (not necessarily irreducible) bridges on these k-strips conditioned to terminate at fixed points $m + ik$, with $m \in \mathbb{Z}$, correspond exactly to the self-avoiding random walk on the k-strip conditioned to terminate at $m + ik$. The purpose of the section is to allow for the use of known numerical algorithms for simulating UPSAWs in the creation new algorithms for simulating the self-avoiding random walk on the k-strip.

In Section 3, we review key properties of $\text{SLE}_{8/3}$ then diagram how properties of $\text{SLE}_{8/3}$ and the conjecture that the fine-mesh scaling limit of SAW is $\text{SLE}_{8/3}$ can be used to predict certain random variables associated with the self-avoiding random walk on the k-strip for $k >> 0$. Finally, in the three subsections of Section 4, we use the conclusions of Section 2 to design numerical simulations that test the predictions made in the three subsections of Section 3, respectively.

2. Half-plane SAWs and bridges

2.1. **Infinite half-plane SAW [Lawler].**

2.2. **Interior point [Lawler].**

2.3. **Limit of counting measure [Lawler].**

3. Predictions using $\text{SLE}_{8/3}$

It was realized in the late 1990's that a stochastic differential equation, now called (chordal) Schramm-Loewner evolution (SLE_κ) in recognition of Oded Schramm's contribution to the theory, characterizes a family of measures on 2-d paths distinguished by their invariance with respect to conformal transformation and a stochastic requirement called the domain Markov property. This differential equation is a special case of the classical Loewner equation of complex analytic origin.

Schramm's collaborations with Werner and Lawler eventually led to a proof that a process supported on square mesh grids in the plane, called the loop-erased random walk (originally defined in [5]), converges weakly to SLE_2 as the mesh size goes to 0 ([6]). The loop-erased random walk is a random walk that avoids itself, but it is not equivalent to UPSAW. In [7], the same authors argue that the following conjecture must hold if one supposes that SAW is a conformally invariant measure. In this conjecture and for the remainder of the chapter we use the notation $\Im(z) :=\text{Im}[z]$ and $\Re(z) :=\text{Re}[z]$.

Conjecture 3.1. *UPSAW converges weakly to $SLE_{8/3}$ as the mesh size goes to* 0*. Moreover, if $f : \mathbb{H} \to \{z \in \overline{\mathbb{H}} : 0 < \Im z < 1\}$ is a conformal map whose unique continuous extension to $\overline{\mathbb{H}}$ satisfies $f(0) = 0$ and $f(\infty) = x + i$, then the self-avoiding random walk on the normalized k-strip, conditioned to terminate at $x + i$, converges weakly to $SLE_{8/3}$ as $k \to \infty$.*

A proof that SAW is indeed a conformally invariant process is currently intractable, so the statement remains a conjecture; however, there is strong numerical evidence to support this conjecture ([1]). An illustration of $\text{SLE}_{8/3}$ generated by a numerical simulation is pictured in Figure 1.

In this section, we use Conjecture 3.1 to predict exact distributions of random variables associated with bridges, ω, conditioned to have large heights, $h(\omega) >> 0$. These distributions are then compared with numerically simulated distributions of bridges that are adapted from simulations of half-planes SAWs by way of the propositions in Section 2.

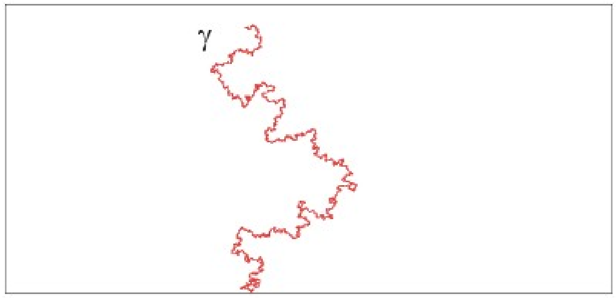

FIGURE 1. Numerical simulation of $\mathrm{SLE}_{8/3}$

3.1. **Density of real component of terminal point for the self-avoiding random walk on the normalized k strip.**

Conjecture 3.2. *Let P^k denote the probability measure of the self-avoiding random walk on the normalized k-strip. Then for all $k > 0$, $P^k[\Im\omega(|\omega|) = 1] = 1$ and*

$$\lim_{k\to\infty} P^k[\Re\omega(|\omega|) \leq x] = \int_{-\infty}^{x} \cosh\left(\frac{\pi}{2}\xi\right)^{-5/4} d\xi$$

3.2. **Rightmost point distribution.** We begin by stating an important result for $\mathrm{SLE}_{8/3}$. The following theorem, originally proved in [7], is crucial in the derivation of the main prediction in this section.

Theorem 3.1. (Lawler, Schramm, Werner) *Let γ be the $SLE_{8/3}$ generating curve. Suppose $A \subset \overline{\mathbb{H}}$ is compact and $\mathbb{H} \setminus A$ is simply connected with $0 \notin A$. If $\Phi_A : \mathbb{H} \setminus A \to \mathbb{H}$ denotes the unique conformal map that fixes 0 and ∞ and has $\Phi_A'(\infty) = 1$, then the distribution of two-dimensional curves given by chordal $SLE_{8/3}$ satisfies,*

$$\mathbf{P}[\gamma \cap A = \emptyset] = \Phi_A'(0)^{5/8}$$

By utilizing Theorem 3.1 and Conjecture 3.1, we can calculate hitting-probabilities for the SAWs in the strip, ω, under the probability measure described above. This is accomplished as follows. Choose $x \in R$ and $y > 0$. We want to compute the probability that a chordal $\mathrm{SLE}_{8/3}$ generating curve, γ, in the y-strip initialized at 0 and terminated at $x + iy$ contains a point $\gamma(t)$ with $\Re\gamma(t) > r > x$. The $\mathrm{SLE}_{8/3}$ process is conformally invariant, so for each choice of $x \in R$ and $r > x$, we can apply the sequence of conformal maps illustrated in Figure 2 and effectively reduce the problem to computing the probability that a chordal $\mathrm{SLE}_{8/3}$ generating curve in the upper half plane (from 0 to ∞) intersects a semicircle, A, on the boundary of $\mathbb{H}$, as seen in the figure. The map Φ_A, defined and pictured in Figure 3, satisfies the antecedent of Theorem 3.1, therefore allowing us to compute the desired probability exactly.

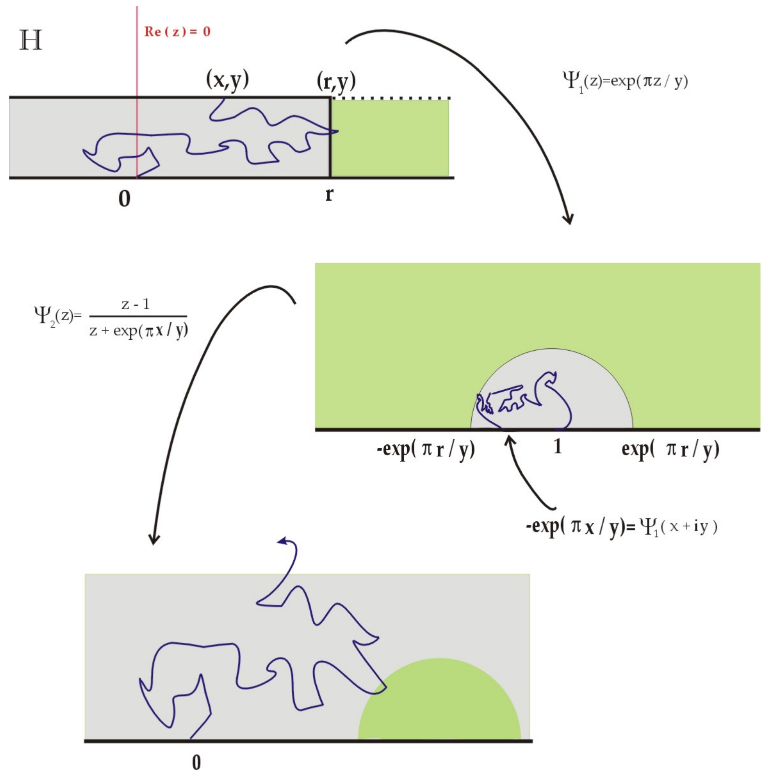

FIGURE 2. A sequence of conformal maps

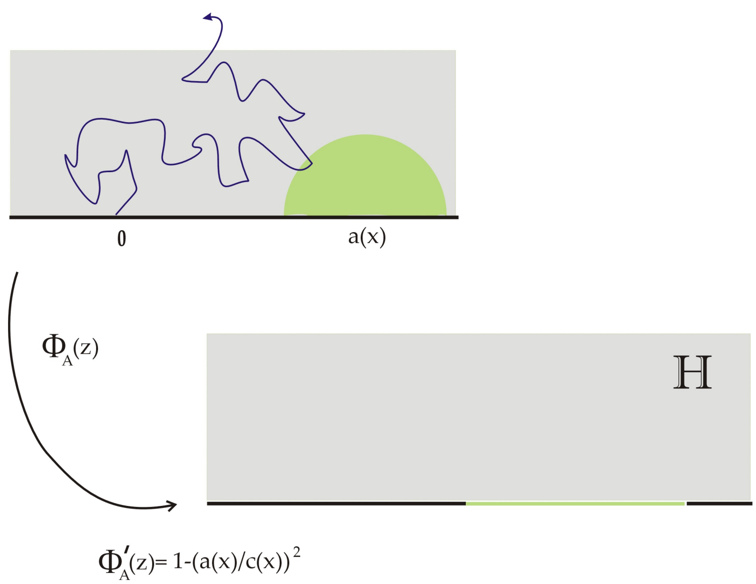

FIGURE 3. Φ map

Next, we use then use Conjecture 3.2 to get the following prediction for $k >> 0$,

$$\mathbf{P}\left[\max_{1\le j\le|\omega|}(\Re(\omega(j)) > r)\right] \approx \int_{\mathbb{R}} \cosh\left(\frac{x\pi}{2k}\right)^{-5/4}\left(1-\left(\frac{a(x)}{c(x)}\right)^2\right)dx.$$

The integral on the right hand side of the previous equation can be evaluated numerically to allow for an approximation of the probability that SAW in the k-strip, with $k >> 0$, has $\max_{0\le j\le|\omega|}\Re(\omega(j)) > r$. In Subsection 4.2, we compare the theoretical rightmost point density conjectured here with the experimental rightmost point density achieved with numerical simulations.

3.3. **Left-passing probability.** One example of a useful computational formula associated with SLE_κ is Schramm's *left-passing probability*, $p(z)$, of a point $z \in \overline{\mathbb{H}}$ with respect to the SLE_κ generating curve (Schramm called this a 'left crossing probability,' but we use slightly different terminology here). This function on $\mathbb{H}$ is defined for $\kappa \in (0,8)$, and its definition is given in terms of winding numbers. For $\kappa \in (0,4]$, an equivalent, more easily stated definition is given by,

$$\mathbf{P}[\gamma \text{ passes left of } z] = p(z) = \mathbf{P}[z \in H_\infty^+] \tag{1}$$

where H_∞^+ is defined to be the connected component of $\overline{\mathbb{H}}\setminus\gamma[0,\infty)$ that contains $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. If $\kappa \in (0,4]$, then γ is simple and $\gamma(t) \to \infty$ w.p.1, so $\overline{\mathbb{H}}\setminus\gamma[0,\infty)$ has exactly two simply connected components; thus, p is well-defined.

It is important to point out that any definition of p depends on the conjecture that $\gamma(t)$ diverges to ∞ as $t \to \infty$, a known property of the SLE_κ generating curve. It is also known that if γ is the SLE_κ generating curve for any $0 < \kappa < 8$, then for all $z \in \mathbb{H}\setminus\{0\}$,

$$\mathbf{P}[z \in \gamma(0,\infty)] = 0.$$

It follows that,

$$\mathbf{P}[\gamma \text{ passes left of } z] = 1 - \mathbf{P}[\gamma \text{ passes right of } z]. \tag{2}$$

Thus, the event 'γ passes right of z' is exactly the event described by 'γ does not pass left of z.'

The SLE_κ left passing probability, p, can also be defined for a more general range, $\kappa \in (0,8)$, but SAW can only correspond to the special case of $\kappa = 8/3$, so we omit the more general definition of left passing probability which is written in terms of winding numbers of $z \in \mathbb{H}$ with respect to a certain closed curve related to the generating curve [9].

Schramm derived an explicit formula for p by applying Ito's formula to a known martingale and setting the drift component to 0 in order to derive a deterministic Fokker-Plank equation for a two-point hitting density related to the left passing probability, $p(z)$. The partial differential equation attained for $p(z) = p(x+\mathbf{i}y)$ is

$$\left(\frac{2x}{|z|^2}\right)\frac{\partial p}{\partial x} - \frac{2y}{|z|^2}\frac{\partial p}{\partial y} + \frac{\kappa}{2}\frac{\partial^2 p}{\partial x^2} = 0, \tag{3}$$

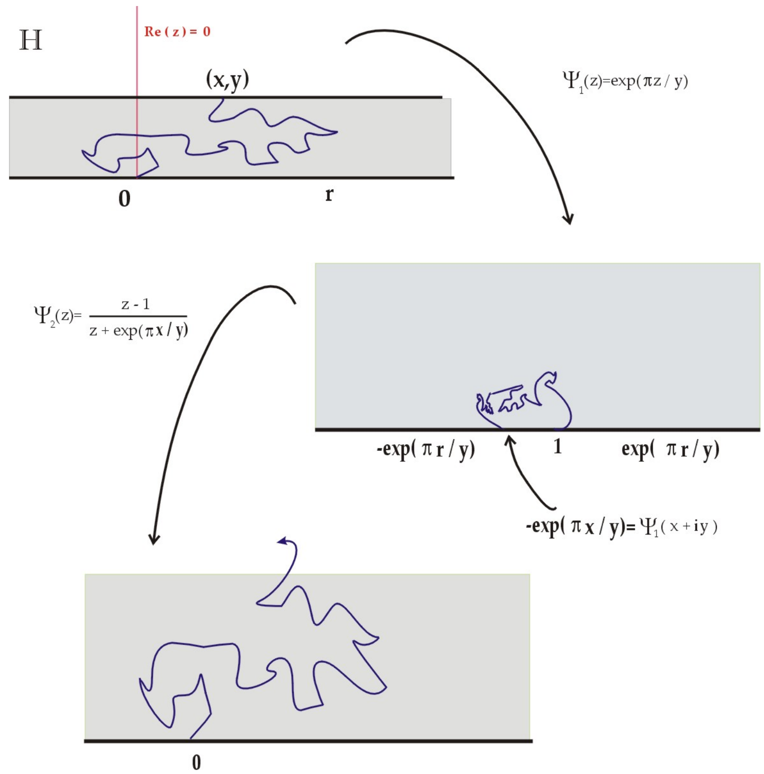

FIGURE 4. A sequence of conformal maps

with boundary conditions $p(z) \uparrow 1$ as $\arg z \downarrow 0$ and $p(z) \downarrow 0$ as $\arg z \uparrow \pi$. The scale invariance of SLE_κ suggests a substitution $w = x/y$ that allows one to reduce (3) to an equation in one variable that can be solved explicitly in terms of hypergeometric functions [9]. In the special case $\kappa = 8/3$, one finds that,

$$\mathbf{P}[\gamma \text{ passes left of } z] = 1 - \cos(\arg(z)) \tag{4}$$

In a similar derivation to that of the previous subsection, we use the fact that $\mathrm{SLE}_{8/3}$ is conformally invariant to define a sequence of conformal maps that allows to compute the left-passing probability of $\mathrm{SLE}_{8/3}$ in the y-strip conditioned to terminate at a fixed point $x + iy$ by using the known formula (4) for $\mathrm{SLE}_{8/3}$ given above. This sequence of conformal maps is defined and illustrated in Figure 4.

Finally, we use then use Conjecture 3.2 to arrive at the following prediction that applies to any point z in the interior of the k-strip ($k >> 0$),

$$\mathbf{P}\left[\text{SAW in the } k\text{-strip crosses left of } z\right] \approx \frac{c}{2}\int_{\mathbb{R}} \cosh\left(\frac{x\pi}{2}\right)^{-5/4}\left(1 - \cos\arg\left(\frac{\exp(\pi z/k) - 1}{\exp(\pi z/k) + \exp(\pi x)}\right)\right) dx.$$

where

$$c = \left(\int_{\mathbb{R}} \cosh\left(\frac{x\pi}{2}\right)^{-5/4}\right)^{-1} \tag{5}$$

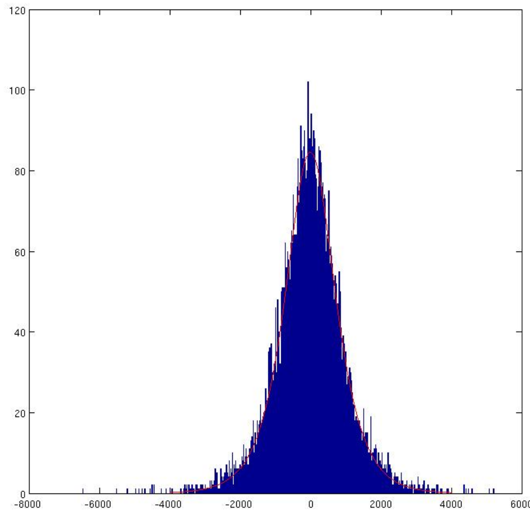

Figure 5. Comparison of numerical data (blue bars) and exact prediction (red line) of the density of the real component of the terminal point for the self-avoiding random walk on the normalized k strip.

This theoretical computation of left-passing probabilities is compared to results of numerical experiments in

4. Numerical Results

The propositions from Section 2 allow us to simulate (not necessarily irreducible) bridges of height k by adapting existing numerical algorithms for simulating infinite half-plane SAWs. This is accomplished by fixing a height k that is much smaller than the expected height of a simulated infinite half-plane SAW with a very a large number of steps, and then collecting all of the bridges of exactly that height.

4.1. **Comparison of theoretical and numerical hitting density.** The comparison is pictured in Figure 5.

4.2. **Comparison of theoretical and numerical rightmost point density.** Comparisons are pictured in Figure 6 and Figure 7.

4.3. **Comparison of theoretical and numerical left-passing probabilities.** A picture of the theoretical left-passing probabilities is shown in Figure 8 and the difference between the theoretical left-passing probabilities and the experimentally computed left-passing probabilities are pictured in Figure 9.

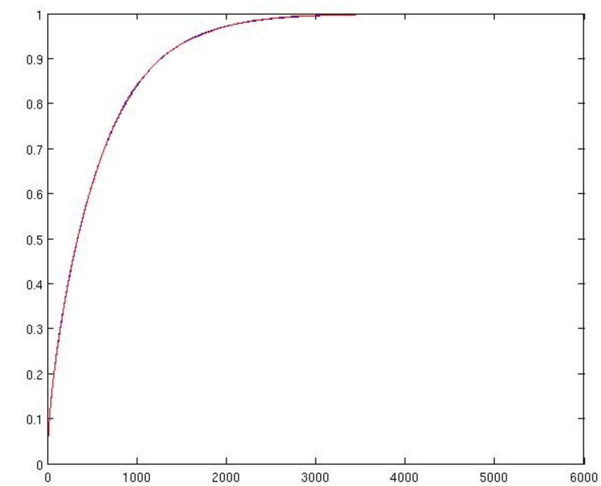

Figure 6. Comparison of the theoretical rightmost point density (blue) and numerical rightmost point density (red).

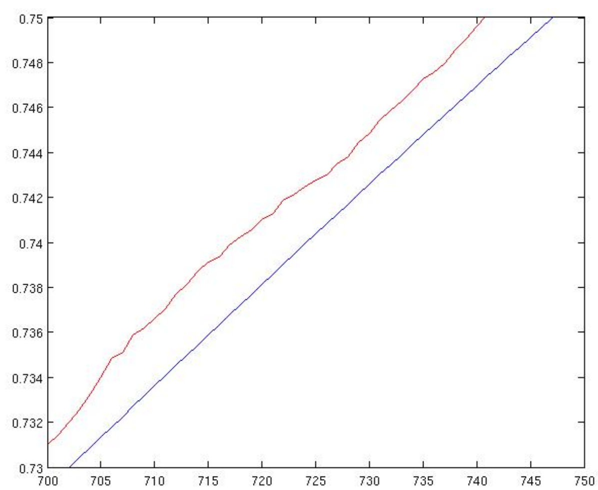

Figure 7. Close-up of comparison of the theoretical rightmost point density (blue) and numerical rightmost point density (red).

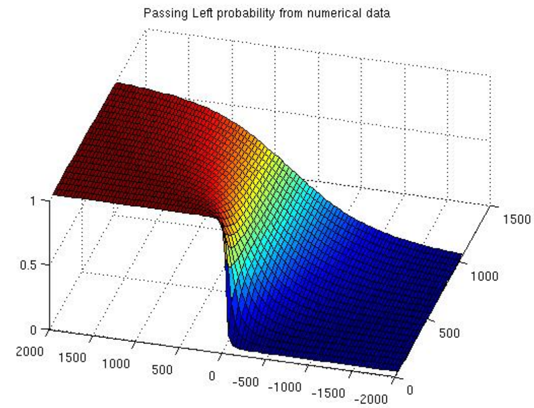

FIGURE 8. Graph of the theoretical left-passing probabilities.

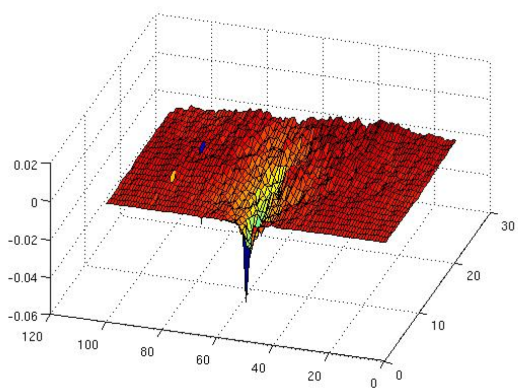

FIGURE 9. Graph of the difference between the theoretical left-passing probabilities and the experimentally computed left-passing probabilities.

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