Exam #1 solutions · Friday, February 2, 2007 · MATH 124 · Calculus I · Section 8

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, except for questions which specifically ask for verbal responses.

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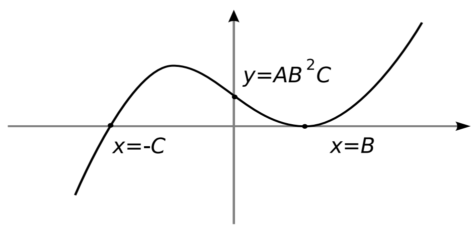
Problem 1. Let $f(x) = A(x - B)^2(x + C)$ with $A > 0$, $B > 0$, and $C > 0$. Using the following space, sketch a graph of $f(x)$. Include and label the x-intercept(s), if any, and the y-intercept. (The x-intercepts and y-intercept should be in terms of A, B, and C — please do not plug in specific numerical values for A, B, and C.)

Solution: The function has a double root at $x = B$ and a single root at $x = -C$. The y-intercept is found by setting $x = 0$, namely, $y = AB^2C$. The leading coefficient is positive and the function is a cubic polynomial, so $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Writing down these things, we have



Problem 2. Sally's track coach tells her that to avoid injury, she should increase her weekly mileage by 10% per week.

Part (a). If she currently runs 2 miles per week, and if she follows her coach's advice, by how many weeks from now will she be running 25 miles per week?

Solution: Sally's current mileage is 2 miles per week. Next week it will be $2 \cdot 1.1$; the week after it will be $2 \cdot 1.1^2$, and so on. After t weeks it will be $2 \cdot 1.1^t$. When will this be 25? Solve for t:

$$\begin{aligned} 2 \cdot 1.1^t &= 25 \\ 1.1^t &= 12.5 \\ \ln(1.1^t) &= \ln(12.5) \\ t\ln(1.1) &= \ln(12.5) \\ t &= \frac{\ln(12.5)}{\ln(1.1)} \approx 26.50. \end{aligned}$$

So, her mileage will surpass 25 miles per week after 27 weeks.

Part (b). If she currently runs 2 miles per week, and if she instead follows her friend's advice and increases her mileage by 1.5 miles per week, by how many weeks from now will she be running 25 miles per week?

Solution: Sally's current mileage is 2 miles per week. Next week it will be $2 + 1.5$; the week after it will be $2 + 1.5 \cdot 2$, and so on. After t weeks it will be $2 + 1.5t$. When will this be 25? Solve for t:

$$\begin{aligned} 2 + 1.5t &= 25 \\ 1.5t &= 23 \\ t &= \frac{23}{1.5} \approx 15.33. \end{aligned}$$

So, her mileage will surpass 25 miles per week after 16 weeks.

Problem 3. Let

$$f(z) = \frac{(z-1)(z-2)}{(z-3)(z-4)}.$$

Part (a). Find the domain of $f(z)$.

Solution: We can evaluate this function for any input unless a division by zero happens. Division by zero happens when the denominator is zero, so we need to solve

$$(z-3)(z-4) = 0$$

to find all z which are not in the domain of $f(z)$. Remember that the product of two real numbers can be zero only when one (or both) of them are zero. So, $(z-3)(z-4) = 0$ when either $z-3 = 0$ or $z-4 = 0$, i.e. $z = 3$ or 4. The domain of $f(z)$ is all other z's, namely, $z \neq 3, 4$. (You may write this in interval notation as $(-\infty, 3) \cup (3, 4) \cup (4, +\infty)$ if you wish. The notation is less important to me than the computation.)

Part (b). Find the zeroes of $f(z)$.

Solution: The zeroes of any function are the input(s), if any, which give zero as output. We find these by setting $f(z) = 0$ and solving for z. We have

$$\begin{aligned} \frac{(z-1)(z-2)}{(z-3)(z-4)} &= 0 \\ (z-1)(z-2) &= (z-3)(z-4) \cdot 0 = 0. \end{aligned}$$

Again, the product of two real numbers is zero when at least one of them is zero, so either $z-1 = 0$ or $z-2 = 0$, i.e. the zeroes of $f(z)$ are $z = 1$ and 2.

Part (c). Find the vertical asymptote(s), if any, of $f(z)$.

Solution: Vertical asymptotes occur when the numerator is not zero but the denominator is zero[1]. In part (a) we found the zeroes of the denominator, namely, $z = 3$ and 4. Neither of these are zeroes of the numerator, so $f(z)$ has vertical asymptotes at $z = 3$ and 4.

Part (d). Find the horizontal asymptote(s), if any, of $f(z)$.

Solution: The horizontal asymptote of a rational function is the ratio of its leading coefficients, namely, $y = 1/1 = 1$. Alternatively, graph the rational function and keeping zooming out in the horizontal direction. Alternatively, evaluate the function for very large positive z and very large negative z and see what the function values approach.

Part (e). Find the hole(s), if any, of $f(z)$.

Solution: By the discussion in part (c), $f(z)$ has no holes. We would need the numerator and the denominator to have a common zero; they do not.

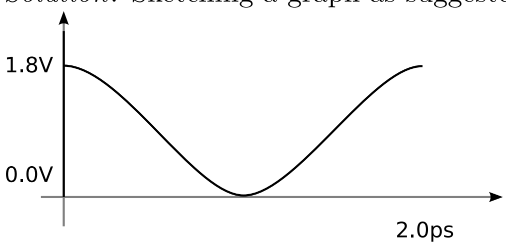
[1]In fact, they occur when the multiplicity of a zero of the denominator exceeds the multiplicitiy of a zero in the numerator. For example, $(z-1)^2/(z-1)^3$ has a vertical asymptote at $z = 1$; $(z-1)^3/(z-1)^2$ has a hole there. However, I've not discussed this generality in class.

Problem 4. In your physics class, you attach an oscillosope to a piece of electronic equipment and discover that the voltage you observe looks like a sine wave. In particular, you notice that

- The maximum voltage you observe is 1.8 volts.
- The minimum voltage you observe is 0.0 volts.
- This maximum voltage is found at the start of your data (i.e. $t = 0.0$ picoseconds); the first time it reoccurs is 2.0 picoseconds later (i.e. $t = 2.0$ picoseconds).

Find an equation of the form $a + b\cos(ct)$ that describes the voltage V as a function of time t in picoseconds. (You don't need to know how many picoseconds make a second.) (Hint: You might want to sketch a graph.)

Solution: Sketching a graph as suggested, we have



From the graph we can read off the following data:

- Amplitude = 0.9V.
- Vertical shift = 0.9V.
- Period = 2.0 ps.
- Horizontal shift = 0.

From data, we can write down an equation:

$$\begin{aligned} V &= \text{vertical shift} + \text{amplitude} \cdot \cos\left(\frac{2\pi}{\text{period}}(t - \text{horizontal shift})\right) \\ &= 0.9 + 0.9\cos\left(\frac{2\pi}{2.0}(t-0)\right) \\ &= 0.9 + 0.9\cos(\pi t). \end{aligned}$$

You can (and should) graph this equation in your calculator to make sure it looks like the given graph.

Problem 5. In a lab experiment, you have microorganisms living in a petri dish. Assume that the microorganisms are dying at an exponential rate. You have the following data:

- You forgot to count the initial population.
- You count a population of 2500 organisms on day one.
- You count a population of 1500 organisms on day two.

Find an exponential model of the form $y = Ce^{rt}$ for these data, where y is the number of microorganisms in the dish after t days. That is, find C and r so this model matches the given data.

Solution: We are given $y = 2500$ at $t = 1$ and $y = 1500$ at $t = 2$. Writing these down, we have

$$\begin{cases} 2500 &= Ce^{r} \\ 1500 &= Ce^{2r}. \end{cases}$$

This is two equations in two unknowns, so we need to eliminate a variable. We can solve for C in one equation and plug it into the other. (Some students prefer to divide one equation by the other. This works too.) From the first equation,

$$C = 2500e^{-r}.$$

Plugging into the second, we have

$$\begin{aligned} 1500 &= 2500e^{-r}e^{2r} \\ e^{r} &= \frac{1500}{2500} = 0.6 \\ \ln(e^{r}) &= \ln(0.6) \\ r &= \ln(0.6) \approx -0.5108. \end{aligned}$$

Since we know $C = 2500e^{-r}$ and $r = -0.5108$, we have

$$C = 2500e^{0.5108} \approx 4166.$$

In summary, we have found

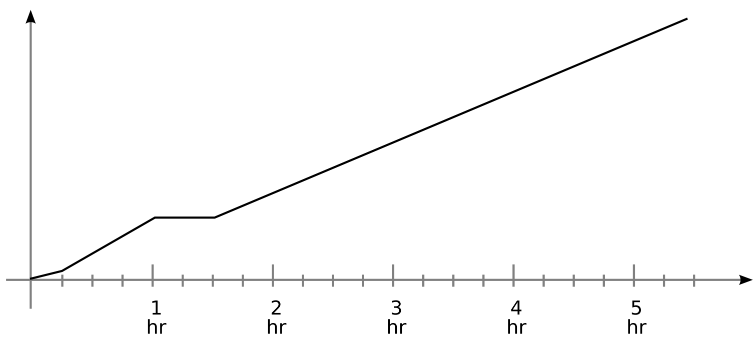
$$y = 4166e^{-0.5108t}.$$

Problem 6. You leave on a car trip for San Diego. The following events happen:

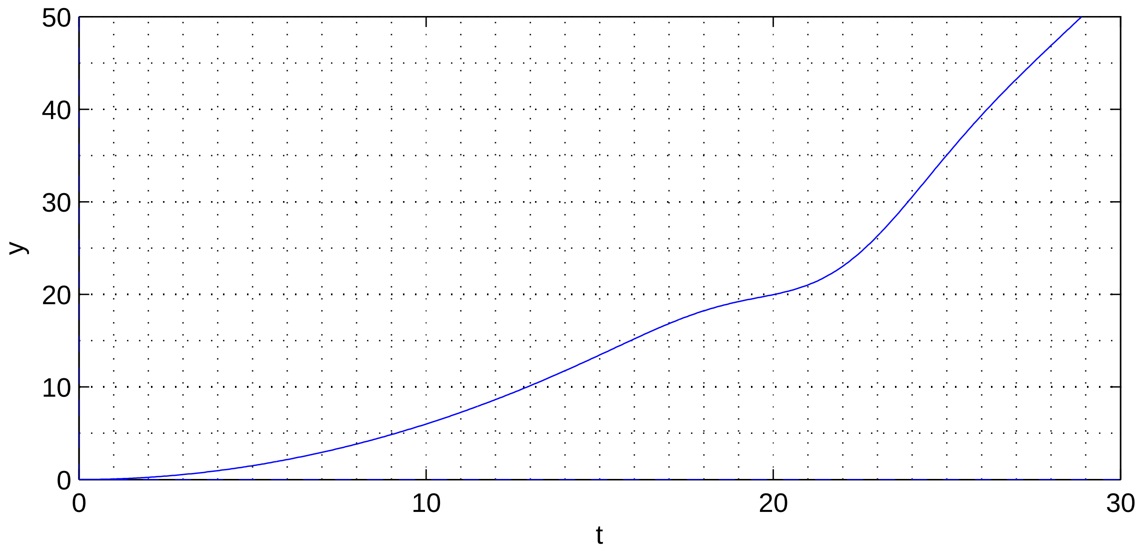
(i) You drive at medium speed (in town) for the first 15 minutes.
(ii) Once you get out of town, you accelerate almost immediately to the speed limit and drive at that speed for 45 minutes.
(iii) Then, you hit a traffic jam. You come to a stop almost immediately and sit at a stop for 30 minutes.
(iv) Once past the accident, you accelerate almost immediately back to the speed limit and drive at that speed until you reach the coast 4 hours later.

Sketch a graph of distance D from your house vs. time t in hours. Label all the above events on the t axis.

Solution:



Problem 7. The following graph shows the altitude y of a rocket (in miles) as a function of time t since launch (in seconds):

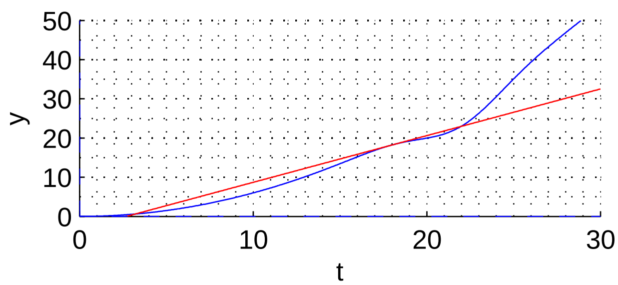


Part (a). Estimate $f(18)$, and describe verbally what it means to you. Include units in your response.

Solution: From the graph, $f(18) \approx 18$. This means that the rocket is 18 miles high after 18 seconds.

Part (b). Estimate $f'(18)$, and describe verbally what it means to you. Include units in your response.

Solution: Sketching a tangent line to $t = 18$ in red, I see something like



The slope of this line looks like about 1.2; let's call it 1. This must be miles per second; it's the (vertical) speed of the rocket after 18 seconds.

Part (c). Estimate $f^{-1}(40)$, and describe verbally what it means to you. Include units in your response.

Solution: From the graph, it looks like $f^{-1}(40) = 26$, i.e. the rocket is 40 miles high after about 26 seconds of flight.

Problem 8. Let $f(x) = 2x^2 + 1$. Evaluate

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

using properties of limits.

Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{[2(a+h)^2 + 1] - [2a^2 + 1]}{h} \\
&= \lim_{h\to 0} \frac{[2(a^2 + 2ah + h^2) + 1] - [2a^2 + 1]}{h} \\
&= \lim_{h\to 0} \frac{[2a^2 + 4ah + 2h^2 + 1] - [2a^2 + 1]}{h} \\
&= \lim_{h\to 0} \frac{2a^2 + 4ah + 2h^2 + 1 - 2a^2 - 1}{h} \\
&= \lim_{h\to 0} \frac{4ah + 2h^2}{h} \\
&= \lim_{h\to 0} (4a + 2h) \\
&= 4a.
\end{aligned}$$

Problem 9. The following are selected values of a function $f(x)$.

$f(3) =$ is undefined $\qquad f(4) = -6 \qquad f(6) = 9$

Table A

x	$f(x)$
2.9	-8.14286
2.99	-98.01493
2.999	-998.00150
2.9999	-9998.00015

Table B

x	$f(x)$
3.1	-12.15789
3.01	-102.01508
3.001	-1002.00150
3.0001	-10002.00015

Table C

x	$f(x)$
3.9	-5.56566
3.99	-5.95070
3.999	-5.99501
3.9999	-5.99950

Table D

x	$f(x)$
4.1	6.57576
4.01	6.05071
4.001	6.00501
4.0001	6.00050

Table E

x	$f(x)$
5.9	6.32184
5.99	6.49616
5.999	6.49956
5.9999	6.49999

Table F

x	$f(x)$
6.1	6.63196
6.01	6.59234
6.001	6.50034
6.0001	6.50002

Table G

x	$f(x)$
100	1.05285
1000	1.00503
10000	1.00050
100000	1.00005

Table H

x	$f(x)$
-100	2.95257
-1000	2.99503
-10000	2.99950
-100000	2.99995

Part (a). Find the following limits, when they exist. If a limit does not exist, say why not. For each, state which table(s) (A, B, C, D, E, F, G, H) you use to support your answer.

- $\lim_{x\to 3} f(x)$: This does not exist. As $x \to 3$ from either side (tables A and B), $f(x)$ goes more and more negative. This looks like a vertical asymptote at $x = 3$.
- $\lim_{x\to 4} f(x)$: The left-hand limit (table C) looks like -6; the right-hand limit (table D) looks like 6. Since those two don't agree, the limit does not exist.
- $\lim_{x\to 6^-} f(x)$: This looks like 6.5 (table E).
- $\lim_{x\to -\infty} f(x)$: From table H, it looks like $f(x) \to 3$ as $x \to -\infty$.

Part (b). Determine if $f(x)$ is continuous or not continuous at the following values. If $f(x)$ is not continuous at the point, describe what feature occurs there.

- $x = 3$: The function isn't defined there, so it can't be continuous there. It looks like a vertical asymptote.
- $x = 4$: The function value and the left-hand limit are -6, which disagrees with the right-hand limit 6. So, the function is not continuous there. It has a jump.
- $x = 6$: The left-hand and right-hand limits are both 6.5, but the function value at $x = 6$ is 9. The function is not continuous there. It has a jump.