Exam #3 solutions · Tuesday, April 3, 2007

MATH 124 · Calculus I · Section 8 · Spring 2007

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, unless a problem specifically requests a verbal response.
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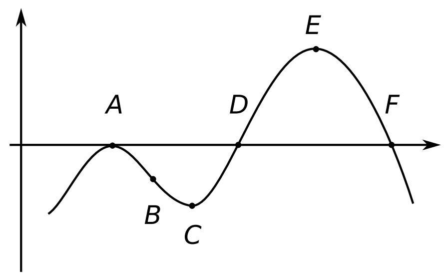
Problem 1. Consider the following graph of $f'(x)$:



I repeat for emphasis: I am showing you the graph of $f'(x)$, but I will ask you questions about $f(x)$.

Part (a). Which of the labeled points are critical points of $f(x)$?

Solution: Critical points of f occur where f' is zero or undefined. There are none of the latter; f' is zero at points A, D, and F.

Part (b). Which of those critical points are local maxima (not minima) of $f(x)$? (In this problem, I am not interested in local extrema at boundary points.)

Solution: Local maxima must occur at the critical points (A, D, and F) or boundary points; the latter are not of interest here. We can use the first-derivative test on each of these three points.

- At point A, $f'(x)$ is negative on both sides of A. This means f is decreasing on either side of A, so A is a false alarm of f.
- At point D, $f'(x)$ is negative to the left and positive to the right. Thus D is a local minimum of f.
- At point F, $f'(x)$ is positive to the left and negative to the right. Thus F is a local maximum of f.

Alternatively, we can try the second-derivative test.

- At point A, $f'(x)$ has zero slope, i.e. $f''(A)$ is 0. The second-derivative test is inconclusive here.
- At point D, $f'(x)$ has positive slope, i.e. $f''(D) > 0$. Thus D is a local minimum of f.
- At point F, $f'(x)$ has negative slope, i.e. $f''(F) < 0$. Thus F is a local maximum of f.

In conclusion, f has a local maximum at F.

Part (c). Which labeled points are inflection points of $f(x)$?

Solution: Inflection points of f occur where f'' is zero, i.e. where f' has zero slope. These are points A, C, and E.

Problem 2. Find dy/dx if $\ln(y)\sin(y) = \cos(x)$.

Solution: Using implicit differentiation and the product rule, we have

$$\begin{aligned}
\frac{1}{y}\sin(y)\frac{dy}{dx} + \ln(y)\cos(y)\frac{dy}{dx} &= -\sin(x) \\
\left(\frac{1}{y}\sin(y) + \ln(y)\cos(y)\right)\frac{dy}{dx} &= -\sin(x) \\
\frac{dy}{dx} &= \frac{-\sin(x)}{\left(\frac{1}{y}\sin(y) + \ln(y)\cos(y)\right)} = \frac{-y\sin(x)}{\sin(y) + y\ln(y)\cos(y)}.
\end{aligned}$$

Problem 3. Let $f(z) = z^4 + a/z^4$. Find a such that $f(z)$ has a minimum at $z = 2$.

Solution: If $f(z)$ is to have a minimum at $z = 2$, we must have $f'(2) = 0$. Differentiating, we have

$$f'(z) = 4z^3 - 4a/z^5;$$

evaluating at $z = 2$ we have

$$f'(2) = 32 - 4a/32.$$

Setting this to zero and solving for a gives us

$$\begin{aligned}
32 - 4a/32 &= 0 \\
32 &= 4a/32 \\
4a &= 1024 \\
a &= 256.
\end{aligned}$$

Problem 4. Let $g(x) = \sin^2(x) - \cos(x)$.

Part (a). Find an equation for the tangent line to $g(x)$ at $x = \pi/2$.

Solution: As usual, we use point-slope form: $y = y_0 + m(x - x_0)$. Here, $x_0 = \pi/2$; $y_0 = g(x_0) = 1$; $m = g'(x_0)$. Using the chain rule we get $g'(x) = 2\sin(x)\cos(x) + \sin(x)$; evaluating at x_0 we have $g'(\pi/2) = 1$. Putting these together we obtain

$$y = 1 + (x - \pi/2) = x + 1 - \pi/2.$$

Part (b). Write down an equation for the error function involving $g(x)$ and the linear approximation.

Solution: The error function is the original function minus the tangent-line approximation:

$$E(x) = (\sin^2(x) - \cos(x)) - (x + 1 - \pi/2) = \sin^2(x) - \cos(x) - x - 1 + \pi/2.$$

Problem 5. Find b and d such that $f(x) = x^3 + bx^2 + d$ has an inflection point at $x = 3$ and y-intercept -5.

Solution: The y-intercept of a function $f(x)$ is $f(0)$. Thus $d = -5$. If f is to have an inflection point at $x = 3$, we must have $f''(3) = 0$. Differentiating, we have

$$\begin{aligned} f'(x) &= 3x^2 + 2bx \\ f''(x) &= 6x + 2b. \end{aligned}$$

Evaluating at $x = 3$ and setting equal to zero, we get

$$\begin{aligned} 6 \cdot 3 + 2b &= 0 \\ 2b &= -18 \\ b &= -9. \end{aligned}$$

Problem 6. Does the limit

$$\lim_{x\to\infty} \frac{x^2}{e^{0.0024x}}$$

exist? If so, what is it, and why? If not, why not?

Solution: This limit is of the form ∞/∞, so l'Hôpital's rule applies. Differentiating top and bottom we get

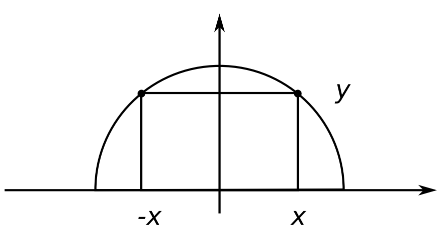
$$\lim_{x\to\infty} \frac{2x}{0.0024e^{0.0024x}}$$

which is still of the form ∞/∞. Using l'Hôpital's rule again, we have

$$\lim_{x\to\infty} \frac{2}{0.0024^2 e^{0.0024x}}.$$

This has a constant on top and a function increasing without bound on the bottom, so the limit is zero.

Problem 7. A rectangular region is to be painted inside a semicircular region as follows:



The semicircle has radius 7 feet. Find x that maximizes the area of the rectangle. (You must show all your work and use calculus.)

Solution: The area of the rectangle is its length times its width, i.e. $A = 2xy$. To be able to do calculus on A we need to write it in terms of a single variable. The equation of a circle of radius 7 centered at the origin is $x^2 + y^2 = 49$; the equation for the upper semicircle is $y = \sqrt{49 - x^2}$. Substituting this into A and differentiating (using the product and chain rules), we have

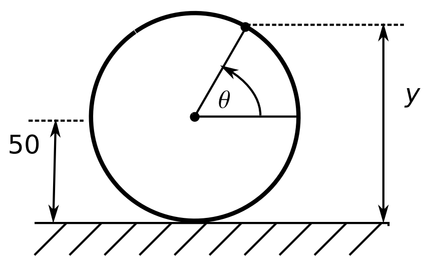
$$\begin{aligned} A &= 2x\sqrt{49 - x^2} \\ \frac{dA}{dx} &= 2\sqrt{49 - x^2} + 2x\frac{-2x}{2\sqrt{49 - x^2}} \\ &= 2\sqrt{49 - x^2} - \frac{2x^2}{\sqrt{49 - x^2}}. \end{aligned}$$

Setting this equal to zero and solving for x we have

$$\begin{aligned} 2\sqrt{49-x^2} - \frac{2x^2}{\sqrt{49-x^2}} &= 0 \\ \sqrt{49-x^2} &= \frac{x^2}{\sqrt{49-x^2}} \\ 49 - x^2 &= x^2 \\ 2x^2 &= 49 \\ x &= 7/\sqrt{2}. \end{aligned}$$

The maximum of A happens at a critical point (which we just found) or a boundary point. The boundary values of x are 0 and 7; areas there are both zero. Thus the maximum area occurs at $x = 7/\sqrt{2} \approx 4.950$.

Problem 8. An amusement-park ride is upright and circular with 50-foot radius. Passengers are seated around the perimeter while the wheel turns counterclockwise at some (as yet unknown) constant rate:



The height of the labeled passenger above the ground is given by

$$y = 50 + 50\sin(\theta).$$

If the passenger's height above ground is increasing by 3.2 feet per second at angle $\theta = \pi/3$, how fast is the wheel turning? Include units in your answer.

Solution: The quantities y and θ both vary with time and are related by the equation $y = 50 + 50\sin(\theta)$. Differentiating both sides with respect to time we find

$$\frac{dy}{dt} = 50\cos(\theta)\frac{d\theta}{dt}.$$

Since we are given the rate of change of y and we are asked to find the rate of change of θ, we need to solve for $d\theta/dt$. This is

$$\frac{d\theta}{dt} = \frac{dy/dt}{50\cos(\theta)}.$$

When $\theta = \pi/3$, this is

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = \frac{3.2}{50\cos(\pi/3)}. = \frac{3.2}{50 \cdot 0.5} = 0.128.$$

The units of $d\theta/dt$ are the units of θ over the units of t, i.e. radians per second.