Exam #4 study guide · Math 124 · Calculus I · Section 8 · Spring 2007

Disclaimers about the study guide:

- Exam 3 covers sections 4.8 through 6.4. While all *topics* on the exam will be taken from this study guide, the specific *questions* on the exam will not be identical to the ones you see here.
- In addition to consulting this guide, please review all homework problems for sections 4.8 through 6.4. In particular, look at unassigned problems nearby. For example, if I assigned #14, see if you can do #13 and #15.
- For reference, you can: (*) use the back of the book; (*) use the student solution manual; (*) make use of the tutor center in Math East 145; (*) ask questions in class; (*) talk to me after class, or during office hours.

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2007-04-19

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Differentiation:

- You still need to know all your differentiation rules, i.e. derivatives of elementary functions, product and quotient rules, chain rule, etc. See the exam 2 and exam 3 study guides for more information.

Parameteric equations:

- Write a parametric equation for a line between two specified points.
- Write a parametric equation for a line through a specified point, in a specified direction.
- Given a curve specified by a parameteric equation and a point on the curve, find an equation for the tangent line to the curve at that point.
- Given a parameteric equation specifying the motion of a particle along a curve, compute the speed of the particle at a given location or time.
- Given a parameteric equation specifying the motion of a particle along a curve, find the time(s) when the particle comes to a stop. (Hint: solve $dx/dt = 0$ and $dy/dt = 0$ as simultaneous equations, i.e. find t such that $dx/dt = 0$ and $dy/dt = 0$.)
- Given parameteric equations specifying the motion of two particles, find the time(s) when the particles collide.

Numerical and graphical integration:

- Intuition for integrals: Extend the familar concept of $D = RT$ to handle non-constant R. We sum up RT products over small T intervals to obtain total distance.
- The integral $\int_a^b f(x)dx$ is interpreted graphically as the signed area between $f(x)$ and the x axis, from a to b. Area below the x axis is taken to be negative.
- Describe the practical interpretation of a definite integral, including units. Example: If $R(t)$ is rate of change of global population in billions of people per year and t is years, then $\int_{2000}^{2004} R(t)dt$ is the population increase from 2000 to 2004 in billions of people.
- Approximate a definite integral using left-hand or right-hand sums, with a specified rectangle width (e.g. $\Delta x = 0.25$) or specified number of rectangles (e.g. $n = 10$). Given input units, give the units of the definite integral: units of x times units of $f(x)$. Be able to do this when the integrand is specified by an equation, a graph, or a data table.
- Remember that velocity is the first derivative of position: $v = dx/dt$.

- Remember that acceleration is the first derivative of velocity: $a = dv/dt$.

Fundamental theorem of calculus:

- To compute definite integrals: $\int_a^b F'(x)dx = F(b) - F(a)$. Point: *If* the integrand is the derivative of something (some function F) *then* we may replace all the rectangle sums etc. with the relatively simple calculation $F(b) - F(a)$.
- For antiderivatives: $F(b) = F(a) + \int_a^b F'(x)dx$. Point: Given a known value of the function at a point and the known derivative of the function, you can find the value of the function at subsequent points.

Average values:

- Definition: $\frac{\int_a^b f(x)dx}{b-a}$.
- Be able to compute an average value when the integrand is specified by an equation, a graph, or a data table.

Properties of definite and indefinite integrals:

- Constant multiples pull out.
- Integrals distribute through sums and differences of functions.
- $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.
- Area between $g(x)$ (above) and $f(x)$ (below) is $\int_a^b (g(x) - f(x))dx$.
- Comparison: If $g(x) \geq f(x)$ for all x between a and b, then $\int_a^b g(x)dx \geq \int_a^b f(x)dx$.
- Use this in particular for upper and lower bounds on functions: $M \geq f(x) \geq m$ for all x in $[a, b]$ means $\int_a^b M dx \geq \int_a^b f(x)dx \geq \int_a^b m dx$. The first integral is just $M(b - a)$ and the last integral is just $m(b - a)$.

Antiderivatives obtained symbolically:

- Know antiderivatives for constant functions, polynomial functions, $1/x$, exponential function, sine and cosine.
- Remember that you always get a family of functions: e.g. $\int \cos x = \sin x + C$.
- Use antiderivatives (indefinite integrals) to compute definite integrals. Example: find $\int_2^8 e^x dx$. Since $\int e^x dx = e^x + C$, we have $\int_2^8 e^x dx = e^8 - e^2$. (What happened to the integration constant? It canceled out: $(e^8 + C) - (e^2 + C) = e^8 - e^2$.)

Antiderivatives obtained graphically:

- Given a qualitative graph (i.e. no axis labels) of $f'(x)$ and a specified value for $f(0)$, sketch $f(x)$.
- If axis labels are given in the graph of $f'(x)$, then be able to specify coordinates of critical points and inflection points of $f(x)$. This requires you to use the FTC from one critical/inflection point to the next. Then, determine concavity of $f(x)$ on various intervals.
- Remember the following:
  - If $f'(x) > 0$ on an interval then $f(x)$ is increasing on that interval.
  - If $f'(x) < 0$ on an interval then $f(x)$ is decreasing on that interval.
  - If $f'(x) = 0$ at a point then $f(x)$ has a critical point at that point.
  - If $f'(x)$ is increasing on an interval then $f(x)$ is concave up on that interval.
  - If $f'(x)$ is decreasing on an interval then $f(x)$ is concave down on that interval.
  - If $f'(x)$ has a maximum or minimum at a point then $f(x)$ has an inflection point there.

Antiderivatives obtained numerically:

- Given a data table for $f'(x)$ and a specified value for $f(0)$, fill in a data table for values of $f(x)$. (Here there isn't enough information for you to determine concavity of $f(x)$.) Here too you need to use the FTC.

Initial-value problems:

- Given $f'(x)$ (which I might write in the form dy/dx) and an initial value for $f(0)$ ($y(0)$, respectively) write down the general solution for the differential equation. Then apply the initial condition to determine the value of the integration constant. Specify the units of your solution.

Second fundamental theorem of calculus:

- Statement: If $f(x)$ is continuous, then

$$\frac{d}{dx}\int_0^x f(t)dt = f(x).$$

- Point: this means that $F(x) = \int_0^x f(t)dt$ is an antiderivative of $f(x)$.
- Given a graph of $f(x)$ and an initial value of $F(x)$, sketch $F(x) = \int_0^x f(t)dt$.
- Given a data table for $f(x)$ and an initial value of $F(x)$, fill in a data table for values of $F(x) = \int_0^x f(t)dt$. Example: $F(0) = 0$ and

x	0	2	4	6	8
$f(x)$	1.1	1.3	1.4	1.2	0.9

First, $F(0) = 0$. Then,

$$\begin{aligned}
F(2) &= F(0) + \int_0^2 f(t)dt \approx 0 + 2\cdot 1.1 = 2.2 \\
F(4) &= F(0) + \int_0^4 f(t)dt = F(2) + \int_2^4 f(t)dt \approx 2.2 + 2\cdot 1.3 = 4.8 \\
F(6) &= F(0) + \int_0^6 f(t)dt = F(4) + \int_4^6 f(t)dt \approx 4.8 + 2\cdot 1.4 = 7.6 \\
F(8) &= F(0) + \int_0^8 f(t)dt = F(6) + \int_6^8 f(t)dt \approx 7.6 + 2\cdot 1.2 = 10.0.
\end{aligned}$$

- Chain-rule problems involving the SFTC: We write

$$F(x) = \int_a^x f(t)dt,$$

and the SFTC says that

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

i.e. $F'(x) = f(x)$. So, it is legitimate to write

$$F(g(x)) = \int_a^{g(x)} f(t)dt$$

and ask about

$$\frac{d}{dx}F(g(x))$$

Using the chain rule, this is

$$\begin{aligned}\frac{d}{dx}F(g(x) &= F(g(x))g'(x) \\ &= f(g(x))g'(x).\end{aligned}$$

Example:

$$\frac{d}{dx}\int_2^{x^3} \sin(t^2)\,dt = \sin(x^6)\cdot 3x^2.$$