Exam #4 solutions · Thurday, April 26, 2007

MATH 124 · Calculus I · Section 8 · Spring 2007

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, unless a problem specifically requests a verbal response.

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Problem 1. Let

$$x(t) = 3t^2 - 6t$$
$$y(t) = \frac{4}{3}t^3 - 4t.$$

Part (a). Find the time(s) t, if any, when the particle comes to a stop.

Solution: We need to find when dx/dt and dy/dt are both zero. We have

$$dx/dt = 6t - 6$$
$$dy/dt = 4t^2 - 4.$$

The former is zero when $t = 1$; the latter is zero when $t = \pm 1$. They are simultaneously zero only at $t = 1$.

Part (b). Find an equation for the tangent line to this curve at $t = 3$.

Solution: Parametric equations for a line through a point (x_0, y_0) in the direction a, b are

$$x = x_0 + at$$
$$y = y_0 + bt.$$

The point (x_0, y_0) is found by plugging $t = 3$ into the original equations:

$$x_0 = 3 \cdot 3^2 - 6 \cdot 3 = 9$$
$$y_0 = \frac{4}{3}3^3 - 4 \cdot 3 = 24.$$

The a and b are $dx/dt|_{t=3}$ and $dy/dt|_{t=3}$:

$$a = \left.\frac{dx}{dt}\right|_{t=3} = 6 \cdot 3 - 6 = 12$$
$$b = \left.\frac{dy}{dt}\right|_{t=3} = 4 \cdot 3^2 - 4 = 32$$

So, the tangent line has equations

$$x = 9 + 12t$$
$$y = 24 + 32t.$$

Problem 2. The function $H(t)$ describes the growth rate in thousands per month of flour beetles in a jar, where t is measured in months since the start of the year.

Part (a). What are the units of $\int_4^7 H(t)dt$?

Solution: The units of an integral are the units of the integrand times the units of the dependent variable. Since $H(t)$ is thousands of beetles per month and t is months, the units of the integral are thousands of beetles.

Part (b). Give a practical interpretation of $\int_4^7 H(t)dt$.

Solution: This is the population change between month 4 and month 7.

Problem 3. The air pressure within a chamber is given by

$$P(t) = 2.1 + 0.4t^{0.5}$$

where P is in units called atmospheres and t is measured in hours. Find the average pressure over the time between $t = 2$ hours and $t = 5$ hours.

Solution: Compute

$$\frac{\int_2^5 P(t)}{5-2} = \frac{\int_2^5 (2.1 + 0.4t^{0.5})}{5-2}.$$

The integral is

$$2.1t + \frac{0.4t^{1.5}}{1.5}\Bigg|_2^5 = \left(10.5 + \frac{0.4 \cdot 5^{1.5}}{1.5}\right) - \left(4.2 + \frac{0.4 \cdot 2^{1.5}}{1.5}\right) \approx 8.53.$$

Then the average value is

$$8.53/3 \approx 2.84.$$

Note that if you graph $P(t)$ from 2 to 5, you'll see that it rises slowly from $P(2) \approx 2.7$ to $P(5) \approx 3.0$. Thus, an average of 2.84 is reasonable.

Problem 4. Find the exact area between $f(x) = e^x - 2$ and $g(x) = -1$ on the interval $[2, 4]$.

Solution: The area between $f(x)$ and $g(x)$ from 2 to 4 is $\int_2^4 (f(x) - g(x))\, dx$ so we can compute

$$\begin{aligned}
\int_2^4 (e^x - 2 - (-1))\, dx &= \int_2^4 (e^x - 1)\, dx \\
&= e^x - x\Big|_2^4 \\
&= (e^4 - 4) - (e^2 - 2) \\
&= e^4 - e^2 - 2.
\end{aligned}$$

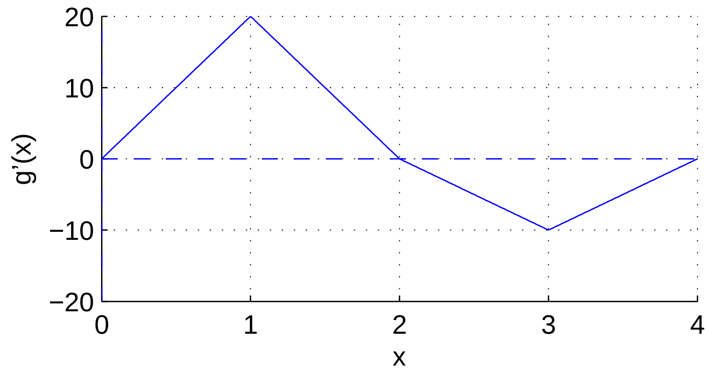
Problem 5. Find the general antiderivative:

$$\int \left(\frac{y^{2.1}}{3} - \frac{7}{y} + 0.2Ae^y + B\right)\, dy.$$

Solution:

$$\frac{y^{3.1}}{9.3} - 7\ln(|y|) + 0.2Ae^y + By + C.$$

Problem 6. Let $g'(x)$ be given by the following graph, and suppose $g(0) = 2$:



Part (a). What are the x-coordinates of the critical points of $g(x)$?

Solution: Critical points of $g(x)$ occur when $g'(x)$ is zero (or undefined). From the graph we see $g'(x) = 0$ at $x = 0, 2, 4$.

Part (b). What are the x-coordinates of the inflection points of $g(x)$?

Solution: Inflection points of $g(x)$ occur at extrema of $g'(x)$. These are at $x = 1, 3$.

Part (c). Find the values of $g(x)$ at the critical and inflection points.

Solution: We are given $g(0) = 2$, and we are given $g'(x)$ in graphical format. This suggests the fundamental theorem of calculus, one landmark at a time. By “landmark” I mean a critical or inflection point.

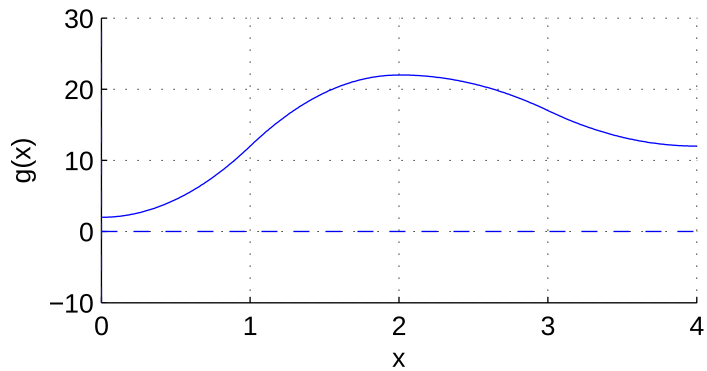
$$\begin{aligned}
g(1) &= g(0) + \int_0^1 g'(x)\,dx = 2 + 10 = 12. \\
g(2) &= g(1) + \int_2^1 g'(x)\,dx = 12 + 10 = 22. \\
g(3) &= g(2) + \int_3^2 g'(x)\,dx = 22 - 5 = 17. \\
g(4) &= g(3) + \int_4^3 g'(x)\,dx = 17 - 5 = 12.
\end{aligned}$$

Part (d). Sketch a graph of $g(x)$. Label critical points and inflection points of $g(x)$.
Solution: Using the above information we can plot the critical points $(0, 2)$, $(2, 22)$, and $(4, 12)$, along with the inflection points $(1, 12)$ and $(3, 17)$. Then we can connect the dots as follows:

- Since $g'(x)$ is positive on $(0, 2)$, $g(x)$ is increasing there.
- Since $g'(x)$ is negative on $(2, 4)$, $g(x)$ is decreasing there.
- Since $g'(x)$ increases on $(0, 1)$ and $(3, 4)$, $g(x)$ is concave up there.
- Since $g'(x)$ decreases on $(1, 2)$ and $(2, 3)$, $g(x)$ is concave down there.

(The sketch is on the next page.)

Here is a sketch:



Problem 7. The quantity A varies with time as specified by

$$\frac{dA}{dt} = 7.3\cos(t) - 0.04.$$

Part (a). Write down a general solution for A.

Solution: The general antiderivative is

$$7.3\sin(t) - 0.04t + C.$$

Part (b). Given that $A(0) = 2.1$, write down a specific solution for A.

Solution: Using the initial condition, we have

$$2.1 = 7.3\sin(0) - 0.04 \cdot 0 + C = C.$$

So, the solution is

$$7.3\sin(t) - 0.04t + 2.1.$$

Problem 8.
Part (a). Let

$$G(x) = \int_1^x e^{t^2}\,dt.$$

Determine (with justification) whether $G(x)$ is increasing, decreasing, or constant.

Solution: A function is increasing, decreasing, or constant when its derivative is positive, negative, or zero, respectively. The derivative is

$$G'(x) = \frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}$$

using the second fundamental theorem of calculus. The output of exponential functions is always positive, so this function is increasing everywhere.
Here is another solution, which a student came up with: e^{t^2} is always positive, so as the right endpoint of the integral slides rightward from 1, only positive areas are added. Thus, the integral function is increasing.

Part (b). Now let

$$G(x) = \int_1^{ax+b} e^{t^2}\,dt.$$

Find $G'(x)$.

Solution: Using the second fundamental theorem and the chain rule, we evaluate the integrand at the right endpoint (upper limit), times the derivative of the right endpoint. This is

$$G'(x) = ae^{(ax+b)^2}.$$