Daniel:

I am unable to decipher a step in the Qmath paper. You write that "a calculation" shows that

$$V(\mathbf{x}, -\mathbf{x}, 0, 0) = \frac{2a}{\|\mathbf{x}\|} + O(a^2).$$

I have

$$V(\mathbf{x}, -\mathbf{x}, 0, 0) = \int \left(1 - e^{-\frac{1}{4}\int_0^{4\beta} U(\omega(s))ds}\right) dW^{4\beta}_{\mathbf{x},-\mathbf{x}}(\omega).$$

I dare not Taylor-expand $1 - e^{-I}$ (where I is the integral) into $I + O(I^2)$ since the integral is infinite in the case of the hard-core potential. I really need the integral to remain exponentiated.

Using the definition of the stochastic integral, I have

$$V(\mathbf{x}, \mathbf{y}, 0, 0) = \int 1 \, dW^{4\beta}_{\mathbf{x},\mathbf{y}}(\omega) - \int e^{-\frac{1}{4}\int_0^{4\beta} U(\omega(s))ds} dW^{4\beta}_{\mathbf{x},\mathbf{y}}(\omega).$$

The first term is known to be $1/g_{4\beta}(\mathbf{x} - \mathbf{y})$; the second is

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} g_{4\beta/n}(\mathbf{x} - \mathbf{z}_1) g_{4\beta/n}(\mathbf{z}_1 - \mathbf{z}_2) \cdots g_{4\beta/n}(\mathbf{z}_{n-2} - \mathbf{z}_{n-1}) g_{4\beta/n}(\mathbf{z}_{n-1} - \mathbf{y}) \\
&\qquad \exp\left\{ \frac{-\beta}{n} \left(\sum_{j=1}^{n-1} U(\mathbf{z}_i) \right) \right\} \, d\mathbf{z}_1 \cdots d\mathbf{z}_{n-1} \\
= &\lim_{n\to\infty} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} g_{4\beta/n}(\mathbf{x} - \mathbf{z}_1) g_{4\beta/n}(\mathbf{z}_1 - \mathbf{z}_2) \cdots g_{4\beta/n}(\mathbf{z}_{n-2} - \mathbf{z}_{n-1}) g_{4\beta/n}(\mathbf{z}_{n-1} - \mathbf{y}) \\
&\qquad \prod_{j=1}^{n-1} \exp\left\{ \frac{-\beta}{n} U(\mathbf{z}_i) \right\} \, d\mathbf{z}_1 \cdots d\mathbf{z}_{n-1}.
\end{aligned}$$

Now, $e^{-(\beta/n)U(\mathbf{z}_i)}$ is 1 when $\|\mathbf{z}_i\| \geq a$ and 0 when $\|\mathbf{z}_i\| < a$. So this becomes

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\|\mathbf{z}_1\|\geq a} \cdots \int_{\|\mathbf{z}_{n-1}\|\geq a} g_{4\beta/n}(\mathbf{x} - \mathbf{z}_1) g_{4\beta/n}(\mathbf{z}_1 - \mathbf{z}_2) \cdots g_{4\beta/n}(\mathbf{z}_{n-2} - \mathbf{z}_{n-1}) g_{4\beta/n}(\mathbf{z}_{n-1} - \mathbf{y}) \\
&\qquad d\mathbf{z}_1 \cdots d\mathbf{z}_{n-1}.
\end{aligned}$$

I don't see how $\mathbf{y} = -\mathbf{x}$ permits any simplification ....

Am I ignorant of some basic fact about Brownian motion?