Referee report on the paper :

Shift in critical temperature for random spatial permutations with cycle weights

by: John Kerl

In this article, the Author considers a model of random spatial permutations on the lattice $\mathbb{Z}^3$ with Boltzmann weights and energy term proportionnal to the length of permutation jumps. This model arises naturally in the description of Bose-Einstein condensation through Feynman-Kac representation of symmetrized N-bosons Hamiltonian, a subject that has generated renewed analytical interests since the nineties.

Here, the interaction energy considered by the Author depends only on cycle lengths and takes the form derived by Betz and Ueltschi, eq. (1.3) (cited refs: [BU07], [BU08]).
This expression involves unknown cycle weights α_l and notwithstanding the apparent simplicity of this interaction, the problem remains a difficult one. So far, only for simplified interaction terms is it possible to get insights about the behaviour of the system. Betz and Ueltschi have studied the cases $\alpha_2 = \alpha$ constant and $\alpha_l = 0$ for $l > 2$. They also solved the case of more general α_l with the restriction that α_l goes to zero in l faster than $1/(\log l)$. For the *Ewens model* considered numerically by the Author, the α_l are kept constant, $\alpha_l = \alpha$ for all l.

Of main interest for this kind of model is the determination of the critical temperature $T_c(\alpha)$ below which longs cycles (i.e. with cycle-length of the order of the system size) appear in the system. In this model of random spatial permutations, this is interpreted as the signal of appearence of Bose-Einstein condensation in the Bose gas according to Feyman's claim.

In his numerical investigations, the Author computes several order parameters: averaged cycle length, fraction of sites in long cycles, correlation length, winding numbers and proportion of sites in winding cycles. Numerics consist in performing Monte Carlo simulations on $3d$ cubic lattices with different sizes, temperatures and couplings. In dimension $d = 3$, all these quantities are expected to exhibit a critical behaviour at some temperature, as was already done when $\alpha = 0$ for the same model.

The main finding of this numerical work is twofold. Firstly, an increase of the critical temperature in the presence of interactions is found and secondly, a linear dependancy of this shift as function of coupling parameter is established, the slope of which matches rather precisely the one found by Betz and Ueltschi in the case of positions varying in the continuum.

As these numerical results are far from being trivial to establish and because the methods used by the Author to get them are sound and confirm certain recent theoretical predictions, I think that this work deserve publication in the JSP.

However, I suggest that the shape and content in which this work is exposed be improved and I would like to comment on this now.

First of all, as is clear from the references, the Author is completing a PhD and wants to explain with a lot of details most of the tools he uses, giving proofs of even obvious facts...

Introduction (Section 1.) states clearly the problem to be solved and is well documented.

Section 2. paragraph 2.3. $\alpha = 0.000$ should be written $\alpha = 0$, there is no need to say that one is considering 0 with 3 precision decimals...

More importantly, in Figure 3 (and this also concerns forthcoming pictures), the existence of a shitft in temperature when coupling is present is not visually obvious. I am pretty sure that higher domain size would impove this visual effect. Of course, this does not call into question the validity of numerical results obtained thereafter.
Is it too difficult to increase L ? How long does it take to make a run with system size $L = 80$? Maybe α could be slightly increased too in order to get better visual evidence...

Section 3. Simulational methods.
In this part, the swap-only algorithm used to simulate the model is explained with explicit analysis of the Markov matrix involved in the Monte Carlo method. Correctness of this algorithm is shown and detailed balance and equilibrium distribution are examined.

This Section should be shortened. For example following definition 3.1 of the distance $d(\pi, \pi')$ between two permutations, do we need a written proof that $d(\pi, \pi')$ is a metric on S_N (Lemma 3.2)? Without loss of clearness, many definitions could be collected together and obvious proofs omitted, like for example in Lemma 3.9 or Corollary 3.10.
In Definition 3.5, the computation of normalization should be shortened.
Proposition 3.17 needs to be formulated in a more precise way and its proof shortened or better, removed.

Section 4. Results.
This Section contains the core of numerical results. The analysis is led correctly and leads to the announced conclusions of this work. Finite size scaling analysis is properly worked out and data collapse (Figure 12.) is significative.
One sees on Figure 9. that system size $L = 80$ leads to correct estimates for critical exponents but as already mentionned, many pictures would produce a better visual effect with higher system size (see remarks for Section 2.)
Here again, better write $\alpha = 0$ in place of $\alpha = 0.000$.

The conclusions 4.7 correctly summarize the results.
Part (2) of conclusion: maybe the Author could comment more on the fact that the lattice structure changes the critical temperature compared to continuum but not the behaviour of the shift when coupling is present.
Part (3): the Author should mention that when coupling $\alpha = 0$, close agreement with Shepp and Lloyd result was already obtained in reference [GRU].

Section 5. Perspectives are interesting but again, should be shortened.