Critical behavior for the model of random spatial permutations

by

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Statement by Author

Dedication

For Steph, who insisted.

Acknowledgments

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Abstract

Critical behavior for the model of random spatial permutations

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We elaborate on a model of random spatial permutations, wherein permutations are weighted according to point positions. This model originates in a study of the interacting Bose gas; the low-temperature-dependent onset of the appearance of arbitrarily long cycles is connected to the phase transition of Bose-Einstein condensates. For our work, we consider a simplified model with point positions held fixed on the cubic lattice, with interactions expressed as Ewens-type weights on cycle lengths of permutations. The critical temperature of the transition to long cycles depends on an interaction-strength parameter α. For weak interactions, the shift in critical temperature is expected to be linear in α with constant of linearity c. Using Markov chain Monte Carlo methods, we find $c = 0.618 \pm 0.086$. This finding matches a similar analytical result of Ueltschi and Betz. We also examine the mean longest cycle length as a fraction of the number of sites in long cycles, recovering an earlier result of Shepp and Lloyd for non-spatial permutations. The plan of this paper is as follows. We begin with a non-technical discussion of the historical context of the project, along with a mention of alternative approaches. Relevant previous works are cited, thus annotating the bibliography. The random-cycle approach to the BEC problem requires a model of spatial permutations. This model it is of its own probabilistic interest; it is developed mathematically, without reference to the Bose gas. Markov-chain Monte Carlo algorithms for sampling from the random-cycle distribution — the swap-only, swap-and-reverse, band-update, and worm algorithms — are presented, compared, and contrasted. Finite-size scaling techniques are used to obtain information about infinite-volume quantities from finite-volume computational data.