Exam #1 solutions · Wednesday, September 17, 2008

MATH 124 · Calculus I · Section 26 · Fall 2008

Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, except for questions which specifically ask for verbal responses.

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Problem 1. Let

$$f(y) = 3y + 7$$

and

$$g(y) = y^2 e^{-2y}.$$

Part (a). Find $g(f(y))$.

Solution:

$$(3y+7)^2 e^{-6y-14}.$$

Part (b).

Solution:

$$y^2 e^{-2y}(3y+7).$$

Problem 2. Solve for z exactly:

$$6 \cdot 3^z = 7 \cdot 2^z.$$

Solution:

$$\begin{aligned}
6 \cdot 3^z &= 7 \cdot 2^z \\
\ln(6 \cdot 3^z) &= \ln(7 \cdot 2^z) \\
\ln(6) + \ln(3^z) &= \ln(7) + \ln(2^z) \\
\ln(6) + z\ln(3) &= \ln(7) + z\ln(2) \\
z(\ln(3) - \ln(2)) &= \ln(7) - \ln(6) \\
z &= \frac{\ln(7) - \ln(6)}{\ln(3) - \ln(2)}.
\end{aligned}$$

Problem 3. An unknown brownish substance weighing 1.5 pounds was placed in your roommate's refrigerator. It was then observed to increase in weight by 7 percent per day. Find the number of days (to the nearest tenth of a day) it will take for the substance to weigh 6 pounds.

Solution:

$$\begin{aligned}
P(t) &= 1.5(1.07)^t \\
6 &= 1.5(1.07)^t \\
4 &= 1.07^t \\
\ln(4) &= \ln(1.07^t) \\
\ln(4) &= t\ln(1.07) \\
t &= \frac{\ln(4)}{\ln(1.07)} \approx 20.5.
\end{aligned}$$

Problem 4. The following data represent profit P (in dollars) versus quantity q of wristwatches sold at a nearby store.

q	P
140	45
180	75
220	105
260	135

Part (a). Write down a linear equation for P as a function of q. Simplify your work as much as possible. Then use the equation to find $P(90)$.

Solution:

$$\begin{aligned} P &= m(q-a)+b \qquad \text{(point-slope form)} \\ a &= 140, b = 45, m = \frac{75-45}{180-140} = \frac{3}{4} \\ P &= \frac{3}{4}(q-140)+45 \\ &= \frac{3q}{4} - 105 + 45 \\ &= \frac{3q}{4} - 60. \\ P(90) &= \frac{3 \cdot 90}{4} - 60 = 67.5 - 60 = 7.5. \end{aligned}$$

Part (b). Verbally describe the significance of the horizontal intercept, from a business point of view.

Solution: A horizontal intercept of a function is an input value which gives zero output. Here, input is q and output is P. The horizontal intercept is the quantity q of wristwatches which result in zero profit. One might think of this as the minimum quantity needed to break even.

Part (c). Find the horizontal intercept.

Solution:

$$\begin{aligned} P &= \frac{3q}{4} - 60 \\ 0 &= \frac{3q}{4} - 60 \\ \frac{3q}{4} &= 60 \\ q &= 80. \end{aligned}$$

Part (d). Verbally describe the significance of the vertical intercept, from a business point of view.

Solution: The vertical intercept of a function is the output which results from a zero input. Here, P is the profit (which may be negative) obtained when zero wristwatches are sold. One may think of this as the overhead cost.

Problem 5. Let

$$f(x) = \frac{x^2 - kx + 3}{x-3}.$$

Find k such that $\lim_{x\to 3} f(x)$ exists.

Solution: As discussed in greater detail in class, we know the denominator goes to zero at $x = 3$. We want to find k so that the numerator also goes to zero at $x = 3$.

$$\begin{aligned} 3^2 - 3k + 3 &= 0 \\ 3k &= 12 \\ k &= 4. \end{aligned}$$

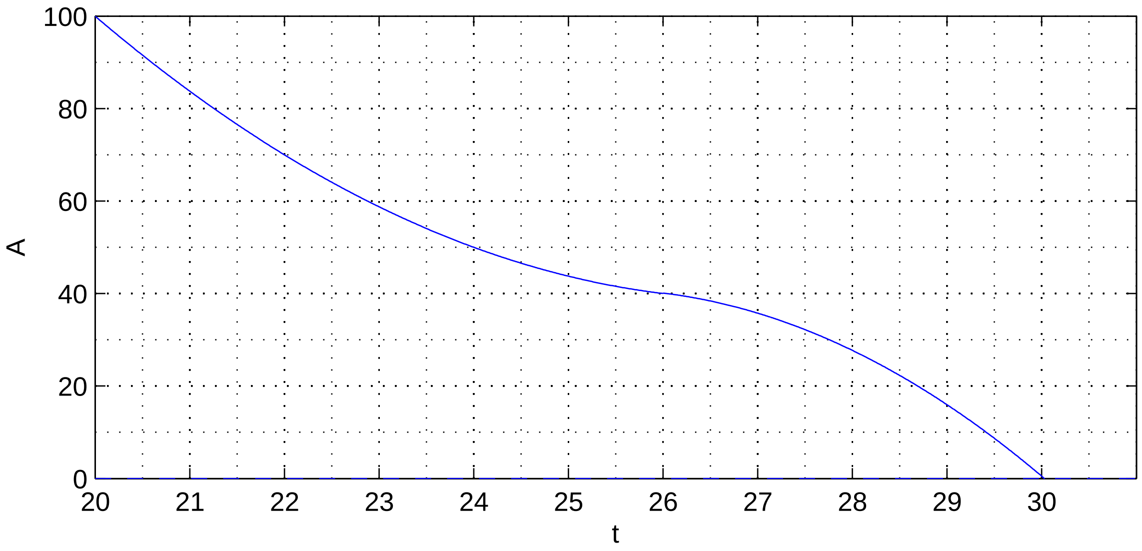
Using this k, we have

$$f(x) = \frac{x^2 - 4x + 3}{x - 3} = \frac{(x-1)(x-3)}{x-3}.$$

The graph of this function is the line $y = x - 1$ with a hole in it at $x = 3$.

I missed an opportunity to make this a two-part question. Part (a) would be as above; part (b) would be to find $\lim_{x\to 3} f(x)$. This would be $x - 1$ evaluated at $x = 3$, namely, 2. Even though the function isn't defined at $x = 3$, we can fimd the limiting value of the function as x *approaches* 3 — which is what limits are all about.

Problem 6. On December 9, 1999 the Mars Polar Lander entered the Martian atmosphere. The following is a reconstruction of its altitude $A(t)$ versus time t in seconds, measured in seconds from the start of the final landing phase. At an altitude of 40 meters, the landing rockets switched off prematurely, letting the lander free-fall to the Martian surface — after which time the lander was never heard from again.



Part (a). Using the graph, estimate $A(24)$. Describe verbally what this quantity means.

Solution: $A(24) \approx 50$. At 24 seconds into the final landing phase, the lander was 50 meters above the Martian surface.

Part (b). Using the graph, estimate $A^{-1}(20)$. Describe verbally what this quantity means.

Solution: $A^{-1}(20) \approx 28.5$. Since A maps seconds to meters, A^{-1} maps meters (here, 20) to seconds (here, 28.5). So, the time when the lander was 20 meters above the surface was 28.5 seconds.

Part (c). Using the graph, estimate the time of impact.

Solution: $A(t) = 0$ at $t \approx 30$ seconds.

Part (d). Using the graph, estimate the lander's speed at time of impact.

Solution: Draw a tangent line to the graph at $t = 30$ and estimate its slope (rise over run). Rise is about -8 meters per run of 0.5 seconds, or -16 meters per second. (This is not a gentle landing.)

In physics courses, velocity is taken to be signed, and speed is taken to be unsigned. That is, speed is absolute value of velocity. I really wanted you to do rise over run and get -16 m/s, and I should have asked for impact velocity. Since I asked for speed, I accepted 16 m/s as an answer.

A few of you said that the speed at impact was 0 meters per second. Based on the way I worded the question, I felt compelled to give you full credit. What I was looking for, though, was the impact speed.

(See the end of this document if you are interested to know where this graph came from.)

Problem 7. Invert the following function:

$$f(x) = Ca^x.$$

(You may assume $C > 0$ and $a > 0$.)

Solution:

$$\begin{aligned}
x &= Ca^y \\
\ln(x) &= \ln(Ca^y) \\
\ln(x) &= \ln(C) + \ln(a^y) \\
\ln(x) &= \ln(C) + y\ln(a) \\
y\ln(a) &= \ln(x) - \ln(C) \\
y &= \frac{\ln(x) - \ln(C)}{\ln(a)} \\
f^{-1}(x) &= \frac{\ln(x) - \ln(C)}{\ln(a)}
\end{aligned}$$

Problem 8. Evaluate the following limit, using algebraic properties of limits:

$$\lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h}.$$

Solution:

$$\begin{aligned}
\lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h} &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\
&= \lim_{h \to 0} 2x + h + 2 \\
&= 2x + 2.
\end{aligned}$$

Problem 9. Numerically estimate the following limit to 3 decimal places:

$$\lim_{\theta \to 0} \frac{2\theta}{\sin(3\theta)}.$$

Solution: Graphing and zooming in repeatedly, one should find 0.667 to three decimal places. (The exact answer turns out to be 2/3, as we will see later in the course when we learn about l'Hôpital's rule.)

Side note: How the graph for question 6 was obtained. This is an illustration of using derivatives to solve for something interesting. My technique uses second derivatives, so this should make sense after we discuss section 2.5.

I read the Wikipedia article on the Mars Polar Lander incident and gathered the following facts:

- The final landing phase (backshell separation) was at 1400 meters altitude, and at that point the lander's velocity was −80 m/s.
- The landing rockets turned off at 40 meters altitude. (They should not have — hence the splat. Apparently a vibration sensor on the landing legs detected vibrations due to descent, and misinterpreted them as having been due to the legs touching the ground.)
- I wanted the 40-meter incident to be visible on the graph so I did a rough sketch of altitude vs. time, from 1400 meters down to zero. Then I kept the part with altitude from 100 meters down to zero. Altitude 100 meters occurred at about 20 seconds.
- The lander's velocity at 12 meters altitude should have been −2.4 meters/second, had the landing rockets not turned off early.
- I arbitrarily used −2.5 m/s as the lander's velocity at 40 meters.
- Under constant acceleration, the height of the lander should be $at^2 + bt + c$. There are two quadratic polynomials, one for each piece of a piecewise function: one for the period when the landing rockets were on, so the net acceleration (landing rockets minus gravity) was positive, and another for the free-fall period when only gravity was involved. That is,

$$A(t) = \begin{cases} a_1t^2 + b_1t + c_1, & 0 \le t \le 26; \\ a_2t^2 + b_2t + c_2, & 26 \le t \le \text{impact time}. \end{cases}$$

  (You didn't need to know these equations to do the exam problem — you just used the graph they produced.)
- Mars acceleration due to gravity (i.e. g) is -3.7 m/s^2. (Compare this to -9.8 m/s^2 on Earth. I'd weigh 55 lbs. on Mars compared to 145 lbs. on Earth. I'd feel unusually light, but a long fall would still hurt.)

Summarizing, I had the following data points.

Before 26 seconds:

- $A(20) = 100$.
- $A(26) = 40$.
- $A'(26) = -2.5$ m/s (velocity at 40 meters).

After 26 seconds:

- $A(26) = 40$ (continuity of height from the other piece of the piecewise function).
- $A'(26) = -2.5$ m/s (continuity of velocity from the other piece of the piecewise function).
- $A''(26) = -3.7$ m/s^2 (acceleration due to Martian gravity).

In both cases, there are three equations in three unknowns. So we can solve for them.

Note that we have

$$\begin{aligned} A(t) &= a_1t^2 + b_1t + c_1 \\ A'(t) &= 2a_1t + b_1 \\ A''(t) &= 2a_1 \end{aligned}$$

for the first piece, and similarly for the second piece (replacing a_1, b_1, and c_1 with a_2, b_2, and c_2).

Before 26 seconds:

$$\begin{array}{rcrcrcrcrcr} A(20) & = & 100: & 400a_1 & + & 20b_1 & + & c_1 & = & 100 \\ A(26) & = & 40: & 676a_1 & + & 26b_1 & + & c_1 & = & 40 \\ A'(26) & = & -2.5: & 52a_1 & + & b_1 & & & = & -2.5 \end{array}$$

After 26 seconds:

$$\begin{array}{rcrcrcrcrcr} A(26) & = & 40: & 676a_2 & + & 26b_2 & + & c_2 & = & 40 \\ A'(26) & = & -2.5: & 52a_2 & + & b_2 & & & = & -2.5 \\ A''(26) & = & -3.7: & 2a_2 & & & & & = & -3.7 \end{array}$$

Eliminating variables, I found

$$\begin{array}{lll} a_1 = 1.25, & b_1 = -67.5, & c_1 = 950; \\ a_2 = -1.85, & b_2 = 93.7, & c_2 = -1145.5. \end{array}$$

Since $2a_1$ and $2a_2$ are the accelerations, we find that the free-fall acceleration was -3.7 m/s^2, as expected, and the acceleration during rocket-assisted descent was 2.5 m/s^2. This is acceleration due to gravity and landing rockets, so we can estimate that the landing rockets themselves were contributing an acceleration of 6.2 m/s^2.