Final Exam Practice

Math 511a

1 Groups

1. Determine all the homomorphisms from S_3 to A_4.
2. Let G be a group of order pqr, where p, q, r are primes and $p > q > r$. Show that G is solvable.
3. Let G be the group of all $n \times n$ invertible matrices over $\mathbb{R}$, $n \geq 3$. Show that G is not solvable.
4. Find all the composition series of the group $\mathbb{Z}/42\mathbb{Z}$. Verify that they are equivalent.
5. Find a central series $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n$ in D_4 such that $G_0 = \{1\}$ and $G_n = D_4$.
6. Give an example of a group G such that G is not nilpotent, but G contains a normal subgroup H such that H and G/H are nilpotent.
7. List all normal subgroups of $A_5 \times A_5$.
8. Suppose S is a set and the symmetric group S_4 acts transitively on S. Determine all possibilities for $|S|$.
9. Show that a group of order 48 must have a normal subgroup of order a power of 2.
10. Let G be the group of real 2×2 matrices of determinant 1, and let H be the subgroup of diagonal matrices.
    (a) Find the normalizer of H in G, $N_G(H)$.
    (b) Find the representatives for the cosets in $N_G(H)$.
11. Let p be a prime number. Let $\mathbb{F}_p$ be the field of p elements. Let $G = GL_2(\mathbb{F}_p)$ be the 2×2 invertible matrices with entries in $\mathbb{F}_p$. Let G act on the vector space $V = \mathbb{F}_p \times \mathbb{F}_p$ in the usual way (by matrix multiplication).
    (a) Show that G has exactly 2 orbits on V.
    (b) Compute the order of the stabilizer of $(1, 0)$.
    (c) Use part (b) to compute the order of G.
12. Either give an example of a finite group having its center of prime index or prove that such a group cannot exist.

13. Suppose p is a prime and G is a finite group. A subgroup K of G is called a normal p-complement if $K \lhd G$ and there is a Sylow p-subgroup P such that $K \cap P = 1$ and $KP = G$. Show that if G has a normal p-complement, then it is unique. Give an example.

14. Let H be the subgroup of S_7, the symmetric group of 7 letters, generated by all 3-cycles. Is the permutation (1234) in H? Explain.

15. Give an example or prove that there does not exist a group of order 5! acting transitively on a set with 9 elements.

16. What are the conjugacy classes of S_3?

17. Suppose G is a group of order 45 with a normal subgroup P of order 3^2. Show that G is abelian. (Hint: $\mathrm{Aut}\,(P)$ has order 6 or 24 according to whether P is cyclic or elementary abelian).

18. True or false: If G is a nonabelian group then it has abelian subgroups H_α such that $G = \cup_\alpha H_\alpha$ and $\cap_\alpha H_\alpha = 1$.

19. Show that the alternating group A_6 has no subgroup of order 72.

2 Rings

1. Determine positive integers n such that $\mathbb{Z}_n$ has no nonzero nilpotent elements.

2. Write the proof if the statement is true; otherwise give a counterexample

   (a) In a ring R, if a and b are idempotent elements, then $a + b$ is an idempotent element.

   (b) In a ring R, if a and b are nilpotent elements, then $a + b$ is nilpotent.

   (c) Every finite ring with 1 is an integral domain.

   (d) There exists a field with seven elements.

   (e) The characteristic of an infinite ring is always 0.

   (f) An element of a ring R which is idempotent, but not a zero divisor, is the identity element of R.

   (g) If a and b are two zero divisors, then $a + b$ is also a zero divisor in a ring R.

   (h) In a finite field F, $a^2 + b^2 = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in F$.

   (i) In a field F, $(a + b)^{-1} = a^{-1} + b^{-1}$ for all nonzero elements such that $a + b \neq 0$.

   (j) There exists a field with six elements.

3. Let R be a ring such that R has no zero divisors. Show that if every subring of R is an ideal of R, then R is commutative.

4. Prove or give counterexample

   (a) There exist only two homomorphisms from the ring of integers into itself.

   (b) The mapping $f : Z \to Z$ defined by $f(n) = 3n$ is a group homomorphism, but not a ring homomorphism.

   (c) The only isomorphism of a ring R onto itself is the identity mapping of R.

   (d) Let R be a ring with 1. Let $f : R \to S$ be a ring homomorphism. Then $f(1)$ is the identity element of S.

   (e) A nonzero homomorphism from a field into a ring with more than one element is a monomorphism.

   (f) Every nontrivial homomorphic image of an integral domain is an integral domain.

5. An idempotent e of a ring R is called a central idempotent if $e \in C(R)$, the center of the ring and $e^2 = e$. Let R be a ring with 1 and e be a central idempotent in R. Show that

   (a) $1 - e$ is a central idempotent in R;

   (b) eR and $(1-e)R$ are ideals of R;

   (c) $R = eR \oplus (1-e)R$

6. Let R be a commutative ring with 1 and $f(x) = a_0 + a_1x + ... + a_nx^n \in R[x]$. If a_0 is a unit and $a_1, a_2, ..., a_n$ are nilpotent elements, prove that $f(x)$ is invertible.

7. Let $f(x) = x^6 + x^3 + 1$. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

8. Give an example of a primitive polynomial which has no root in $\mathbb{Q}$ but is reducible over $\mathbb{Z}$.

9. Show that a proper ideal I of a ring R is a maximal ideal if and only if for any ideal A of R either $A \subseteq I$ or $A + I = R$.

10. Let $f(x) = x^5 + 12x^4 + 9x^2 + 6$. Show that the ideal $I = (f(x))$ is maximal in $\mathbb{Z}[x]$.

11. The ring $R = \mathbb{Q}[x] / \langle x^4 - 16 \rangle$ is a direct sum of fields. Describe the fields explicitly and determine how many of each appear as direct summands.

12. Let $f : R \to S$ be a homomorphism of commutative rings. Prove that $I \subset S$ is a prime ideal, then $f^{-1}(I)$ is also a prime ideal. Give an example where I is maximal but $f^{-1}(I)$ is not maximal.

3 Fields

1. Let E be a field extension of the field F with $[E : F] = p$, where p is a prime. Show that for any element $a \in E \backslash K$ we have $E = K(a)$. Hint: Study the subfields of E.

2. (i) Let F be a field and a, b be members of a field containing F. Suppose that a and b are algebraic of degree m and n over F and $(m, n) = 1$. Show that $[F(a, b) : F] = mn$. (ii) Show this is not necessarily true if $(m, n) \neq 1$.

3. Consider the unique factorization domain $F[t]$, where F is a field and t is transcendental over F. Show that the polynomial $x^2 + tx + t \in F(t)[x]$ is irreducible over $F(t)$. Also show that $x^2 + tx + t \in F(x)[t]$ is irreducible over $F(x)$.

4. Find the splitting field for the following polynomials over $\mathbb{Q}$.

   (i) $x^4 + 1$, (ii) $x^6 + x^3 + 1$

5. Find a splitting field S of $x^4 - 10x^2 + 21$ over $\mathbb{Q}$. Find $[S : \mathbb{Q}]$ and a basis for the splitting field over $\mathbb{Q}$.

6. If F is a field with a finite number of element, prove that F is not algebraically closed.

7. Let $f(x) = x^n - 1 \in \mathbb{Q}[x]$. Show that the Galois group of $f(x)$ over $\mathbb{Q}$ is commutative.

8. Find all proper subfields of $\mathbb{Q}\left(\sqrt[3]{2}, \sqrt{3}, i\right)$.

9. Show that the Galois group of $f(x) = x^3 - 5$ over $\mathbb{Q}$ is isomorphic to S_3.

10. Determine the degree of the extension $\mathbb{Q}\left(\sqrt{3 + 2\sqrt{2}}\right)$