1 Math 511a - Test 1 Practice Questions

Groups

1. Determine all the homomorphisms from S_3 to A_4.

2. Let G be a group of order pqr, where p, q, r are primes and $p > q > r$. Show that G is solvable.

3. Let G be the group of all $n \times n$ invertible matrices over $\mathbb{R}$, $n \geq 3$. Show that G is not solvable.

4. Find all the composition series of the group $\mathbb{Z}/42\mathbb{Z}$. Verify that they are equivalent.

5. Find a central series $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n$ in D_4 such that $G_0 = \{1\}$ and $G_n = D_4$.

6. Give an example of a group G such that G is not nilpotent, but G contains a normal subgroup H such that H and G/H are nilpotent.

7. List all normal subgroups of $A_5 \times A_5$.

8. Suppose S is a set and the symmetric group S_4 acts transitively on S. Determine all possibilities for $|S|$.

9. Show that a group of order 48 must have a normal subgroup of order a power of 2.

10. Let G be the group of real 2×2 matrices of determinant 1, and let H be the subgroup of diagonal matrices.

    (a) Find the normalizer of H in G, $N_G(H)$.

    (b) Find the representatives for the cosets in $N_G(H)$.

11. Let p be a prime number. Let $\mathbb{F}_p$ be the field of p elements. Let $G = GL_2(\mathbb{F}_p)$ be the 2×2 invertible matrices with entries in $\mathbb{F}_p$. Let G act on the vector space $V = \mathbb{F}_p \times \mathbb{F}_p$ in the usual way (by matrix multiplication).

    (a) Show that G has exactly 2 orbits on V.

    (b) Compute the order of the stabilizer of $(1, 0)$.

    (c) Use part (b) to compute the order of G.

12. Either give an example of a finite group having its center of prime index or prove that such a group cannot exist.

13. Suppose p is a prime and G is a finite group. A subgroup K of G is called a normal p-complement if $K \lhd G$ and there is a Sylow p-subgroup P such that $K \cap P = 1$ and $KP = G$. Show that if G has a normal p-complement, then it is unique. Give an example.

14. Let H be the subgroup of S_7, the symmetric group of 7 letters, generated by all 3-cycles. Is the permutation (1234) in H? Explain.

15. Give an example or prove that there does not exist a group of order 5! acting transitively on a set with 9 elements.

16. What are the conjugacy classes of S_3?

17. Suppose G is a group of order 45 with a normal subgroup P of order 3^2. Show that G is abelian. (Hint: $\mathrm{Aut}\,(P)$ has order 6 or 24 according to whether P is cyclic or elementary abelian).

18. True or false: If G is a nonabelian group then it has abelian subgroups H_α such that $G = \cup_\alpha H_\alpha$ and $\cap_\alpha H_\alpha = 1$.

19. Show that the alternating group A_6 has no subgroup of order 72.