Math 511B - Final Exam Practice

1. Let R be a PID and $A \in R^{n \times n}$ representing a map $R^n \xrightarrow{A} R^n$.

   (a) Show that $\operatorname{coker}(A) = 0$ if and only if $\det(A) \in R^{\times}$ and $\operatorname{coker}(A)$ is a torsion R-module if and only if $\det(A) \in R - \{0\}$. ($\operatorname{coker}(A) = R^n / \operatorname{Im}(A)$, i.e. it measures how close to surjective A is just as kernel measure how close to injective a map is)

   (b) Assume $\det(A) \in R - \{0\}$. Let J be the principal ideal $(\det(A))$ in R, and $I := \operatorname{Ann}_R(\operatorname{coker}(A))$. Show that there exists a positive integer N for which

   $$I^N \subset J \subset I.$$

   (c) Assume $R = \mathbb{F}[x]$ for some field $\mathbb{F}$ and that $\det(A) \neq 0$. Show that $\dim_{\boldsymbol{F}} \operatorname{coker}(A)$ is the degree (in x) of the polynomial $\det(A)$.

2. Let R be a PID and let $A \in R^{m \times n}$, $B \in R^{l \times m}$ represent maps of free R-modules as shown, with $BA = 0$ and $m \geq l, n$:

   $$R^n \xrightarrow{A} R^m \xrightarrow{B} R^l.$$

   Let A, B be equivalent by left and right multiplication by invertible matrices over R to diagonal matrices

   $$A \tilde{} \begin{bmatrix} D \\ 0 \end{bmatrix}, \; B \tilde{} \; [E \; 0]$$

   where $D \in R^{n \times n}$ is diagonal with diagonal entries $(d_1, ..., d_n)$ and $E \in R^{l \times l}$ is diagonal with diagonal entries $(e_1, ..., e_l)$. Express the quotient R-module $H := \ker B / \operatorname{Im} A$ as a direct sum of cyclic R-modules in terms of the data given by the d_i, $e_j's$.

3. Let $\mathbb{F}$ be a field and $\mathbb{K}$ a field extension of $\mathbb{F}$ with $[\mathbb{K} : \mathbb{F}] = n < \infty$. Let $f(x)$ be irreducible of degree m and $\gcd(m, n) = 1$. Show that $f(x)$ remains irreducible in $\mathbb{K}[x]$.

4. Let $\mathbb{F}$ be a field and $\mathbb{K}$ a field extension of $\mathbb{F}$ with $[\mathbb{K} : \mathbb{F}] = n < \infty$. LEt k be an integer $0 \leq k \leq n$, and conside the set

   $$\mathbb{G}_{\mathbb{F}}(k, \mathbb{K}) := \{k\text{-dimensional } \mathbb{F}\text{-linear subspaces of } \mathbb{K}\}$$

   (a) Show that the groups of units $\mathbb{K}^{\times}$ acts on the set $\mathbb{G}(k, \mathbb{K})$ in the following way: if V is an $\mathbb{F}$-linear subspace of $\mathbb{K}$ and $\alpha \in \mathbb{K}^{\times}$ then explain why $\alpha V := \{\alpha v : v \in V\}$ is another $\mathbb{F}$-linear subspace of the same dimension. Also explain exactly how this induces an action of the quotient group $\mathbb{K}^{\times} / \mathbb{F}^{\times}$ on $\mathbb{G}(k, \mathbb{K})$.

   (b) Show that if $\gcd(k, n) = 1$, the action of $K^{\times} / F^{\times}$ on $G(k, K)$ is free, i.e. for every non-identity element $\bar{\alpha}$ of the group $K^{\times} / F^{\times}$ there are no k-dimensional F-subspaces $\bar{\alpha} V = V$.

5. Let L be the following matrix in $\mathbb{Z}^{n\times n}$: $(n \in \mathbb{Z})$

$$L = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

   (a) Compute the Smith Normal form for L.

   (b) Express $\mathbb{Z}^n / \operatorname{Im} L$ as a direct sum of cyclic groups.

6. For the matrix A shown below, find the rational canonical form and the Jordan canonical form in $\mathbb{Q}^{4\times 4}$.

$$\begin{bmatrix} 1 & 4 & -4 & 0 \\ -1 & -3 & 2 & 2 \\ 0 & -2 & 1 & 2 \\ -1 & -4 & 2 & 3 \end{bmatrix}.$$

7. Let $G = \mathbb{Z}^3 / (\mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3)$ where v_1, v_2, and v_3 are given by

$$v_1 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}, v_3 = \begin{bmatrix} -6 \\ -8 \\ 14 \end{bmatrix}.$$

   Express G explicitly as a direct sum of cyclic groups.

8. Suppose that E and F are finite Galois extensions of a field k, with E and F both contained in a common extension L of k. Which of the following extensions of k are necessarily Galois extensions of k (sketch a proof or provide a counterexample):

   (a) The compositum EF of E and F (I think we used $E \vee F$)

   (b) $E \cap F$

9. **Solution 1** *Suppose that K/k is a finite Galois extension and that $\alpha_1, ..., \alpha_n$ are distinct elements of K. Assume further that the polynomial $f(x) = (x-\alpha_1)\cdots(x-\alpha_n)$ has coefficients in k. Show that $f(x)$ is irreducible of k if and only if the natural operation of* $\operatorname{Gal}(K/k)$ *on* $\{\alpha_1, ..., \alpha_n\}$ *(by conjugation) is transitive.*

10. Let p be a prime number $\neq 2$, and let ζ be a complex pth root of 1 $(\zeta \neq 1)$. Set $\alpha = \zeta + \zeta^{-1}$. Show that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ and determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

11. Let A be a PID. For which $a \in A$ is the A-module A/aA semisimple?

12. Prove that the Jacobson radical of a semisimple ring is 0.

13. (a) Show that the center of a semisimple ring contains no non-zero nilpotent element. (b) Let $R = kG$ where G is a finite group and k is a field. Assume that the characteristic of k is a prime number p and the p divide the order of G. Prove that R is *not* semisimple. (It may help to consider $\sum_{g \in G} g \in R$).

14. Let V and W be finite dimensional k-vector spaces. Let $V^* = \mathrm{Hom}\,(V, k)$ be the linear dual of V. Given $(\phi, w) \in V^* \times W$, we define a linear map $V \to W$ by $v \to \phi\,(v) \cdot w$. Show that this association defines a bilinear map $V^* \times W \to \mathrm{Hom}\,(V, W)$, and that the induced homomorphism $V^* \otimes W \to \mathrm{Hom}\,(V, W)$ is an isomorphism of k-vector spaces.

15. Let L/K be a finite Galois extension, and let $G = \mathrm{Gal}\,(L/K)$. Let W be a finite dimensional vector space over K, and let $V = W \otimes_K L$ be the associated L-vector space. Note that the map $w \to w \otimes \to$ identifies W with a subset of V. For each $g \in G$, show that there is a bijection $\lambda_g : V \tilde{\to} V$ which satisfies $\lambda_g\,(w \otimes a) = w \otimes ga$ and which is such that $\lambda_g\,(cv) = g\,(c)\,\lambda_g\,(v)$ for $v \in V$ and $c \in L$. (We say that λ_g is "g-linear.") Prove that
$$W = \{v \in V | \lambda_g\,(v) = v \text{ for all } g \in G\} \tag{1}$$

16. Let K be a finite extension of $\mathbb{Q}$ for which $[K : \mathbb{Q}]$ is odd. Show that among the field embeddings $\sigma : K \to \mathbb{C}$, there is at least one which maps K into $\mathbb{R}$. If $K/\mathbb{Q}$ is an odd degree Galois extension, show that all σ map K into $\mathbb{R}$.

17. In the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, prove that $1 \otimes i + i \otimes 1$ is not of the form $x \otimes y$.

18. Let
$$0 \to A \to B \to C \to 0 \ (*)$$
be an exact sequence of R-modules, and let M be an R-module.
    (a) Show by example that the induced sequence
    $$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0 \ (**)$$
    need not be exact.
    (b) Show that $(**)$ is exact if $(*)$ is split.
    (c) Show that $(**)$ is exact if M is projective.
    (d) Show that $(**)$ is exact if $R = \mathbb{Z}$ and $M = \mathbb{Q}$.

19. Write down all abelian groups of order 1500 using both elementary divisors and invariant factors.

20. Let G be a finite cyclic group of order n. Let $\mathbb{Z}G$ denote the integral group ring of G. Show that as rings $\mathbb{Z}G \cong \mathbb{Z}\,[x] \,/\, (x^n - 1)\,\mathbb{Z}\,[x]$. (You may assume that every element of $\mathbb{Z}\,[x] \,/\, (x^n - 1)\,\mathbb{Z}\,[x]$ is uniquely represented as a polynomial of degree less than or equal to $n - 1$ mod $x^n - 1$.)

21. Let R be a commutative ring and let $I \subseteq R$ be an ideal. Considering I as an R-module, show that if $\sum_{i=1}^{n} a_i \otimes b_i = 0$ in $I \otimes_R I$ we have $\sum_{i=1}^{n} a_i b_i = 0$ (find a bilinear map $I \times I \to R$ and use it to get the result).

22. (a) Let R be a commutative ring. Let M be a noetherian R-module and let N be a submodule of M. Show that M/N is noetherian.

    (b) Suppose that a commutative ring S, when regarded as an additive abelian group, is a finitely generated $\mathbb{Z}$-module. Show that S is a noetherian ring.

    (c) Give an example of a noetherian ring that is not finitely generated as a $\mathbb{Z}$-module.

23. (a) Let L/K be a finite separable extension of fields. Show that there exists a finite Galois extension F/K with $L \subseteq F$.

    (b) Let K be a characteristic p, where p is a prime. Let $n \geq 1$. Determine all diagonalizable matrices $M \in GL_n(K)$ such that $M^p = I$.

24. (a) Give an example of an exact sequence of abelian groups $0 \to A \to B \to C \to 0$ and an abelian group N such that the sequence

    $0 \to \operatorname{Hom}(C, N) \to \operatorname{Hom}(B, N) \to \operatorname{Hom}(A, N) \to 0$

    is not exact.

    (b) Find a nonzero abelian group N such that for every exact sequence $0 \to A \to B \to C \to 0$ of abelian groups, the sequence $0 \to \operatorname{Hom}(C, N) \to \operatorname{Hom}(B, N) \to \operatorname{Hom}(A, N) \to 0$

    is exact. Briefly explain why N has this property.

25. (a) Let M and N be free modules over an integral domain R. Let $m \in M$ and $n \in N$. Suppose $m \otimes n = 0$ in $M \otimes_R N$. Show that either $m = 0$ or $n = 0$.

    (b) Give an example of modules M, N over some ring R and elements $0 \neq m \in M$ and $0 \neq n \in N$ such that $m \otimes n = 0$ in $M \otimes_R N$.

26. (a) Determine the similarity class (i.e. the possible rational cannonical forms) for all 3×3 rational matrices $A \in M_3(\mathbb{Q})$ whose characteristic polynomial is $(t-1)^3$. What is the minimal polynomial of each similarity class.

    (b) Find the Jordan form of the 20×20 matrix N which has (i) $N_{i,i} = 2$ for all i and (ii) zeros everywhere else except for the first row and 20^{th} column which is a 1.

27. Let R be a commutative ring with identity and suppose I is an ideal contained in every maximal ideal of R.

    (a) Let $s \in I$, show $1 + a$ is a unit in R. (Hint: consider the ideal $(1 + a)$.)

(b) If M is an R-module, let $IM = \{\sum a_j m_j | a_j \in I, m_j \in M\}$. Show if M is finitely generated and $M = IM$, then $M = 0$. (hint: Consider a minimal set of generators of M.)

28. Suppose $0 \to K \xrightarrow{i} P \xrightarrow{\pi} M \to 0$ and $0 \to K' \xrightarrow{i'} P' \xrightarrow{\pi'} M \to 0$ are exact sequences of R-modules with P projective.

    (a) Show that there exist maps $\alpha : K \to K'$ and $\beta : P \to P'$ such that the diagram

$$\begin{array}{ccccccc} 0 \to & K & \xrightarrow{i} & P & \xrightarrow{\pi} & M & \to 0 \\ & \downarrow_{\alpha} & & \downarrow_{\beta} & & \downarrow_{\text{id}} & \\ 0 \to & K' & \xrightarrow{i'} & P' & \xrightarrow{\pi'} & M & \to 0 \end{array}$$

    commutes.

    (b) If $\theta : K \to P \oplus K'$ is defined by $\theta(x) = (ix, \alpha x)$ and $\psi : P \oplus K' \to P'$ is defined by $\psi(u, v) = \beta(u) - i'(v)$, then

    $0 \to K \xrightarrow{\theta} P \oplus K \xrightarrow{\psi} P' \to 0$

    is exact. Verify the exactness at $P \oplus K'$.

    (c) Conclude that if P' is also projective, then $K \oplus P' \cong P \oplus K'$.

29. If $f : A \to A$ is an R-module homomorphism such that $ff = f$, then $A = \ker f \oplus \operatorname{Im} f$.

30. R is a commutative ring. A is an R-module.

    (a) If $f : A \to B$ is an R-homomorphism, then $f(Tor(A)) \subseteq Tor(B)$, where Tor is torsion submodule.

    (b) If $0 \xrightarrow{f} A \xrightarrow{g} B \to C$ is an exact sequence of R-modules, then so is $0 \to Tor(A) \to Tor(B) \to Tor(C)$ by f_T and g_T respectively.

31. Describe all semisimple rings of order 144.

32. Determine the abelian group $G = (a, b : 30a = 42b = 70(a + b) = 0)$ as a direct sum of cyclic groups.

33. If D is a division ring show that all elements, with one exception are quasi-regular. What is the exception?

34. Determine the Galois group over $\mathbb{Q}$ of $f(x) = x^3 - 3x + 1$.

35. If $R = 2\mathbb{Z}$, the ring of even integers, show that the ideal $I = (6)$ is modular but the ideal $J = (4)$ is not modular.

36. True or false

    (a) A simple Artinian ring is left noetherian.

    (b) The radical of a ring is a radical ring.

37. If A is a finitely generated $\mathbb{Z}$-module, describe $R \otimes_{\mathbb{Z}} A$ as completely as possible.

38. Describe all semisimple rings having 10,000 elements.

39. Let M be $\mathbb{C}^3$ with elements considered as column vectors. We make M into a $\mathbb{C}[x]$ module by having x act by left multiplication by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}$$

and by having elements of $\mathbb{C}$ act by scalar multiplication. Find the rank and torsion of this module and give its decomposition as a direct sum of cyclic modules.

40. Determine all canonical forms and invariants of the following matrix over $\mathbb{F}_4 = \{0, 1, t, 1+t \mid 1+t+t^2 = 0\}$.

$$\begin{pmatrix} 1 & t & t+1 \\ t+1 & 1 & t \\ t & t+1 & 1 \end{pmatrix}$$