August 2001
Algebra Qualifying Exam
Sample Solutions

1A) Let R be the $\mathbb{C}$-subalgebra of $M_2(\mathbb{C})$ (2×2 matrices) generated by the matrix

$$A = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}.$$

(i) What is the $\mathbb{C}$-dimension of R?
(ii) Compute A^{100}.
Answer: (i) We know that the dimension of $M_2(\mathbb{C}) = 4$ and so we have answer between 1 and 4. Let us compute the characteristic and minimal polynomial for A.

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

And therefore $f(A) = A^2 - 2A + 1$ is both the characteristic and minimal polynomial for A. And as $A^2 = 2A - 1$ we know that we only need the identity matrix and A so the dimension is 2.
(ii) To compute high powers we would like to diagonalize the matrix if possible. To do this we need distinct eigenvectors. However I believe we do not have that in this case. But there is a nice pattern to the product as seen below:

$$\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -8 & 5 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 2 \\ -8 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ -12 & 7 \end{bmatrix}$$

And so by induction we have that

$$A^{100} = \begin{bmatrix} -199 & 100 \\ -400 & 201 \end{bmatrix}$$

1B) Consider the linear operator $L = (d/dx)^2$ acting on the vector space $\mathbb{F}_3[x]/(x^{10})$ by formal differentiation ($\mathbb{F}_3$ is the finite field with 3 elements). Find the minimal polynomial of L.
Answer: Let us first consider what the matrix of this linear operator looks like by considering the action on the basis. $1 \to 0$, $x \to 0$, $x^2 \to 2$, $x^3 \to 0$, $x^4 \to 0$, $x^5 \to 2x^3$, $x^6 \to 0$, $x^7 \to 0$, $x^8 \to 2x^6$, $x^9 \to 0$. A matrix for this is

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 2&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&2&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&2&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

We also know that the minimal polynomial divides the characteristic polynomial of x^{10} which we know as this is a lower triangular matrix. We see that $x^2 = 0$ and that is the answer.

2A) Let M be the abelian group of all operatornametions on $\mathbb{Z}/5\mathbb{Z}$ to $\mathbb{Q}$ (the group structure is given by $(f+g)(x) = f(x) + g(x)$). Describe explicitly the automorphism group of the group M.
Answer: Consider the operatornametion that takes $\bar{z} \longmapsto \alpha_z$ for $0 \le z \le 4$ with $\alpha_z \in \mathbb{Q}$. Consider any other operatornametion as $\bar{z} \longmapsto \beta_z$ for $\beta_z \in \mathbb{Q}$. Then any automorphism is an invertible map that takes $\alpha_z \to \beta_z$. By this we have $GL_5(\mathbb{Q})$, all invertible 5×5 matrices over $\mathbb{Q}$.

2B) Let A and B be finite nonabelian simple groups. Determine all normal subgroups of the direct product $A \times B$.

Answer: As A and B are simple nonabelian we know that only 1 and $A \lhd A$ and $1, B \lhd B$. So we know that $A \times 1 \lhd A \times B$, $1 \times B \lhd B$ and $A \times B \lhd A \times B$. If we project $A \times B$ onto A and B by

$$\begin{aligned} f &: A \times B \to A \text{ by } f(a,b) = a \\ g &: A \times B \to B \text{ by } f(a,b) = b \end{aligned}$$

then the correspondence theorem tells us that: If $f : G \to H$ is onto with $\ker f = K$, then $L \longleftrightarrow f^{-1}(L) = \{x \in G : f(x) \in L\}$ is a 1-1 correspondence between the set of all subgroups L of H and the set of all subgroups of G that contain K. Furthermore, $L \lhd H$ if and only if $f^{-1}(L) \lhd G$. So if $N \lhd A \times B$ then $f_A(N) \lhd A$ and $g_B(N) \lhd B$. You can show that $\ker f_A \cap \ker g_B = 1 = A \cap B$ and thus the only normal subgroups are the ones that we have claimed above.

3A) Let R be the ring of C^∞ operatornametions on the real line. Let I_0 and I_1 be the ideals consisting of operatornametions that vanish at 0 and 1 respectively. Give an explicit description of $R/\left[(I_0)^2 \cap I_1\right]$.
Answer: We will use the chinese remainder theorem in this problem as $I_0^2 \oplus I_1 = R$. Therefore we will have that

$$R/\left(I_0^2 \cap I_1\right) \cong R/I_0^2 \oplus R/I_1$$

Now we figure out what each of the direct summands is.
Define a homomorphism as follows:

$$\varphi : R \longrightarrow \mathbb{R} \text{ via } \varphi(f) = f(1).$$

This map is clearly onto. We know that $\ker \varphi = I_1$. Thus by the fundamental homomorphism theorem we know that

$$R/I_1 \cong \mathbb{R}.$$

Now we note that $f \in I_0^2$ implies that $f(0) = 0 = f'(0)$ (just use product rule $f \cdot f' + f' \cdot f = 0 + 0 = 0$). Define a ring structure on $\mathbb{R}^2$ via $\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ bc + ad \end{pmatrix}$ and pointwise addition. Now define

$$\phi : R \longrightarrow \mathbb{R} \text{ via } \phi(f) = (f(0), f'(0))$$

We show that this is a ring homomorphism (the additive part is clear)

$$\begin{aligned} \phi(fg) &= \left(f(0)g(0), (fg)'(0)\right) \\ &= \left(f(0)g(0), f'g(0) + fg'(0)\right) \\ &= \left(f(0), f'(0)\right)\left(g(0), g'(0)\right) \\ &= \phi(fg) \end{aligned}$$

The kernel of this onto map is I_1^2 and we thus have that

$$R/I_0^2 \cong \mathbb{R}^2 \text{ under multiplication given}$$

Thus we have that

$$R/\left[I_0^2 \cap I_1\right] \cong \left(\mathbb{R}^2\right)_{\text{with funny structure}} \oplus \mathbb{R}$$

3B) Let K be any field and let $f = \sum_{i=0}^n a_i x^i \in K[x]$ be a polynomial of degree n. Show that f is irreducible if and only if $g = \sum_{i=0}^n a_{n-i} x^i$ is irreducible.
Answer: Proof by contrapositive. Show f is reducible if and only if g is reducible. Assume that f is reducible. Then $f = hk = (h_0 + h_1 x + \cdots + h_m x^m)\left(k_0 + \cdots + k_{n-m}^{n-m} x^{n-m}\right)$ for some h and k. Then just look at $g = (h_m + \cdots + h_m x^m)\left(k_{n-m} x^{n-m} + \cdots + k_0\right)$ and therefore it is reducible.

4A) Compute the Galois group over $\mathbb{Q}$ of the splitting field (in $\mathbb{C}$) of $f(x) = x^5 - 3$.
Answer: First we note that $f(x)$ is irreducible by the Eisenstein criterion with $p = 3$. This implies that the Galois group G must be a transitive subgroup of S_5. We know that $f(x)$ has 1 real root and 4 complex roots by simple calculus. We know that $x^5 - 3$ can be reduced over $\mathbb{Q}\left(\sqrt[5]{3}\right)$ and we can factor as

$$x^5 - 3 = (x - \alpha)\left(x^4 + \alpha x^3 + \alpha^2 x^2 + \alpha^3 x^1 + \alpha^4\right), \text{ where } \alpha = 5^{1/3}.$$

The remaining irreducible polynomial splits over $\mathbb{Q}(\alpha,\omega)$ where ω are the 5^{th} roots of unity. Thus $[\mathbb{Q}(\alpha,\omega):\mathbb{Q}(\alpha)][\mathbb{Q}(\alpha):\mathbb{Q}] = 4\cdot 5 = 20$. Thus $|G| = 20$. We know that we have an element in the Galois group of order 5 and of order 4. The degree 4 extension of 5^{th} roots of unity is a normal extension but the degree 5 extension of fifth roots of 3 is not normal. As the 2-Sylow subgroups correspond to an extension that is not normal, there must be 5 2-Sylow subgroups. However the 5-Sylow subgroup corresponds to a normal extension and thus we have (up to isomorphism) a $\mathbb{Z}_5 \lhd G$. We know that the group is not abelian as a two cycle does not commute with a 5-cycle. We have an action of $\mathbb{Z}_4$ on $Aut(\mathbb{Z}_5)$ and thus the structure of G is given as a semidirect product with $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$.

4B) Suppose that $f(x) \in \mathbb{Q}[x]$, $g(x) = f(x^2)$, $K \subseteq \mathbb{C}$ is a splitting field for $g(x)$, and $[K:\mathbb{Q}]$ is odd. Show that $f(x)$ and $g(x)$ have the same Galois group.
Answer: Let L be the splitting field for $f(x)$. Then we know that $L \subseteq K$. If we know that roots of $f(x)$ are a_i in the splitting field then the roots of $g(x)$ in its splitting field are $\pm\sqrt{a_i}$. Each of these possible splitting fields are either dimension 1 or 2 as they are quadratic extensions. Therefore we know that

$$\text{odd} = [K:\mathbb{Q}] = [K:L][L:\mathbb{Q}] = 2^k[L:\mathbb{Q}]$$

Thus $k = 0$ and we have the same splitting field for $g(x)$ and $f(x)$ and therefore the same Galois group.

5A) Show that $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}$ and $\mathbb{Q}$ are isomorphic as $\mathbb{Z}$-modules.
Answer: We need to give a $\mathbb{Z}$-module homorphism that is an isomorphism. Consider the map $\varphi:\mathbb{Q}\to\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}$ by mapping $q \longmapsto 1\otimes q$. Then

$$\varphi(q+r) = 1\otimes(q+r) = 1\otimes q + q\otimes r = \varphi(q)+\varphi(r).$$

Also we know that for $z\in\mathbb{Z}$ we have

$$\varphi(zq) = 1\otimes zq = z\otimes q = z(1\otimes q) = z\varphi(q).$$

Now we check that the map is 1-1 and onto. Assume that $\varphi\left(\frac{r}{s}\right) = \varphi\left(\frac{t}{u}\right)$. Then we know that $1\otimes\frac{r}{s} = 1\otimes\frac{t}{u} \implies 1\otimes\frac{r}{s} - 1\otimes\frac{t}{u} = 0 \implies 1\otimes\left(\frac{r}{s}-\frac{t}{u}\right) = 0 \implies \frac{r}{s}-\frac{t}{u} = 0 \implies \frac{r}{s} = \frac{t}{u}$. For surjectivity, given $\frac{a}{b}\otimes\frac{c}{d}$ we need to find an element $q\in\mathbb{Q}$ such that $\varphi(q) = \frac{a}{b}\otimes\frac{c}{d}$. We note that $\frac{a}{b}\otimes\frac{c}{d} = \frac{a}{b}\otimes\frac{c}{d}\frac{b}{b} = \frac{ab}{b}\otimes\frac{c}{db} = 1\otimes\frac{ac}{bd}$. Thus we let $q = \frac{ac}{bd}$.

5B) Let T be the $\mathbb{Q}$-algebra of 3×3 matrices generated by the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & -3 \\ -1 & 0 & -3 \end{bmatrix}$$

Write $T\otimes_{\mathbb{Q}}\mathbb{R}$ as a product of fields.
Answer: We compute the minimal polynomial of A and get $x^3 - 10x + 8$. Therefore we know that $T \cong \mathbb{Q}[x]/\langle x^3-10x+8\rangle$. We now note that $\mathbb{Q}\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{R}$ and also $\mathbb{Q}[x]\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{R}[x]$. Lastly we know that $\mathbb{Q}[x]/\langle x^3-10x+8\rangle\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{R}[x]/\langle x^3-10x+8\rangle$. Using some basic calculus and college algebra we see that $x^3-10x+8$ has 3 distinct real roots, call them α,β,γ. Thus by the chinese remainder theorem we know that

$$\begin{aligned}\mathbb{R}[x]/\langle x^3-10x+8\rangle &\cong \mathbb{R}[x]/\langle x-\alpha\rangle\oplus\mathbb{R}[x]/\langle x-\beta\rangle\oplus\mathbb{R}[x]/\langle x-\gamma\rangle \\ &\cong \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R} = \mathbb{R}^3.\end{aligned}$$