January 1994
Algebra Qualifying Exam

1A) Find all (real) c such that the system

$$\begin{aligned} 2x + c\,(c-1)\,y + 3z &= c+1 \\ -2x + c\,(c-1)\,y - 3z &= c-1 \end{aligned}$$

has a solution, and find the dimension of the space of solutions if they exist.

1B) Find the characteristic polynomial, minimal polynomial, rational canonical form, and Jordan canonical form of

$$A = \begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 8 \\ 0 & -1 & 0 \end{bmatrix}$$

2A) Suppose S is a set and the symmetric group S_4 acts transitively on S. Determine all possibilities for $|S|$.

2B) If $p \in \mathbb{Z}$ is a prime, determine all groups of order $2p$.

3A) Suppose R is an ID (with 1) having only finitely many ideals. Prove that R is a field. What if R is just a commutative ring, not a domain?

3B) Describe all semisimple rings having $10,000$ elements.

4A) Suppose F, K, and L are fields with $F \subseteq K \subseteq L$ and $[L : F]$ finite. Either prove or give a counterexample for each of the following 3 assertions.

(a) If L is Galois over F then L is Galois over K.
(b) If L is Galois over F then K is Galois over F.
(c) If L is Galois over K and K is Galois over F then L is Galois over F.

4B) Suppose F and K are fields with $F \subseteq K$ and $a \in K$ is algebraic over F with $[F(a) : F]$ odd. Show that $F\left(a^2\right) = F(a)$.

5A) Let M be the $\mathbb{Z}$-module $\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})$. Give a precise and explicit description of the ring $End_{\mathbb{Z}}(M)$.

5B) Suppose A and B are finite abelian groups each having all Sylow subgroups cyclic; view A and B as $\mathbb{Z}$-modules. Calculate $A \otimes_{\mathbb{Z}} B$ and determine its Sylow subgroups.