January 1995
Algebra Qualifying Exam

1A) If $A = \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$, find an orthogonal matrix M that diagonalizes A.

1B) Suppose V is a finite dimensional vector space and $T : V \to V$ is a linear transformation for which every nonzero vector is an eigenvector. Prove that T is a scalar multiple of the identity transformation.

2A) Suppose p is a prime and G is a finite group. A subgroup K of G is called a normal p-complement if $K \triangleleft G$ and there is a Sylow p-subgroup P such that $K \cap P = 1$ and $KP = G$. Show that if G has a normal p-complement then it is unique. Give an example.

2B) Let n be a positive integer and $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with complex entries. If $A \in M_n(\mathbb{C})$, denote its determinant by A. Let $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A \neq 0\}$ and let $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : \det A = 1\}$. Show that $GL_n(\mathbb{C})$ is a group under matrix multiplication, that $SL_n(\mathbb{C})$ is a normal subgroup of $GL_n(\mathbb{C})$ and identify the quotient group $GL_n(\mathbb{C}) / SL_n(\mathbb{C})$. You may use basic properties of matrices without deriving them.

3A) Determine the Galois group (over $\mathbb{Q}$) of $f(x) = x^5 + 3x^3 - 2x^2 - 6$.

3B) Say whether each of the following is true or false. Give a proof or counterexample.
(a) Let K be a field of characteristic 0 and let L be an extension of degree 2. Then L is Galois over K.
(b) Let K be a field of characteristic 0 and let L be an extension of degree 3. Then L is Galois over K.

4A) A commutative ring R with 1 is said to be a local ring if it has exactly one maximal ideal M. Prove that every element of R is either a unit or an element of M.

4B) A commutative ring R is called Boolean if $x^2 = x$ for all $x \in R$.
(a) Show that in a Boolean ring $2x = 0$.
(b) Prove that in a Boolean ring then each prime ideal $P \neq R$ is maximal.

5A) Let R be a commutative ring and A an R-module. Let

$$TorA = \{a \in A : \exists r \neq 0 \in R \text{ such that } ra = 0\}$$

(a) If $f : A \to B$ is an R-homomorphism then show that $f(Tor(A)) \subseteq Tor(B)$.
(b) If $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of R-modules, then so is $0 \longrightarrow TorA \xrightarrow{f_T} TorB \xrightarrow{g_T} TorC \longrightarrow 0$ by the maps f and g restricted to the torsion submodules.
(c) If $g : B \to C$ is an epimorphism give an example to show that $g_T : TorB \to TorC$ need not be an epimorphism.

5B) True or false:
(a) Every submodule of a free module is free.
(b) R is commutative with 1; M an R-module implies that M is a finite set if and only if finitely generated and every element is a torsion element.