January 1996
Algebra Qualifying Exams

1A) Let V be a vector space of dimension n over $\mathbb{R}$ and let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = V$ be a sequence of subspaces with $\dim V_i = i$ for all i. Define a group G by

$$G = \{g \in GL(V) : gV_i \subseteq V_i \text{ for all } i\}$$

Give as complete a description as possible of the structure of G.

1B) Show that every 2×2 real matrix with all positive entries can be diagonalized over $\mathbb{R}$.

2A) Find all groups of order 33.

2B) Either give an example of a finite group having its center of prime index or prove that such a group cannot exist.

3A) The ring $R = \mathbb{Q}[x] / \langle x^4 - 16 \rangle$ is a direct sum of fields. Describe the fields explicitly and determine how many of each appears as direct summands.

3B) (i) Proof or counterexample: Every UFD is a PID.
(ii) Give an explicit example (with justification) of an irreducible polynomial of degree 100 in $\mathbb{C}[x, y]$.

4A) Find the Galois group over $\mathbb{Q}$ of $f(x) = x^5 - 80x + 2$. (Hint: How many real roots does $f(x)$ have?).

4B) Let $f(x) = x^4 + 5x^2 + 9 \in \mathbb{Q}[x]$. Is $f(x)$ irreducible? Determine its Galois group over $\mathbb{Q}$.

5A) Proof or counterexample: Every submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module is free.

5B) Let A be the abelian group with presentation:

$$A = \langle a, b, c : 2a + 4b + 2c = 2a + 10b + 8c = 0 \rangle$$

Determine the order and structure of the torsion subgroup T of A.