January 1998
Algebra Qualifying Exams

1A) Let F be a field, let A be an $n \times n$ matrix over F, and let $w \in F^m$ be a column vector. Show that exactly one of the following holds:

(i) $Av = w$ for some $v \in F^n$.

(ii) There is a u in F^m such that $u^T A = 0$ and $u^T w = 1$ (where u^T is the row vector obtained by transposing u).

1B) Let A be the matrix $\begin{pmatrix} 4 & 5 & 3 \\ -5 & -10 & -10 \\ 3 & 6 & 6 \end{pmatrix}$. Compute its characteristic polynomial, minimal polynomial, Jordan canonical form, and rational canonical form.

2A) Let G be a group that acts on a 10-element set S. Suppose that $g \in G$ has order 35. Show that for some positive natural number $n < 35$, the element g^n fixes all points of S.

2B) Let p and q be primes with $p < q$. Show that if there is a non-abelian group of order pq, then $q \equiv 1 \bmod p$.

3A) Let R be a PID. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be an increasing sequence of ideals in R. Prove that the sequence is eventually constant, i.e. for some n, $I_n = I_{n+1} = I_{n+2} = \cdots$.

3B) Determine whether or not the rings $\mathbb{Q}[x]/(p) \oplus \mathbb{Q}[x]/(q)$ and $\mathbb{Q}[x]/(pq)$ are isomorphic where $p = x^4 + 4$ and $q = x^4 + 2^3 - 4x - 4$.

4A) Let p be a prime, and let F be the field with p elements. Suppose that $f \in F[x]$ is a polynomial of degree 4 such that f and $x^{p^2} - x$ are relatively prime. Show that f is irreducible.

4B) Let K be a finite extension of $\mathbb{Q}$ containing primitive n-th roots of unity, and let $\bar{K}$ be an algebraic closure of K. If $b^n = a \in K$ for some $b \in \bar{K} \backslash K$, then show that the extension $K(b)$ over K is Galois with cyclic Galois group. Give a generator of this Galois group explicitly.

5A) Let A be an $n \times n$ rational matrix. Suppose that $d \neq 0$ is a natural number such that the entries of the matrices $\{A^K : K \geq 0\}$ are integral multiples of $1/d$. Show that for some invertible matrix C, the matrix $C^{-1}AC$ has integer entries. Hint: Consider the $\mathbb{Z}$-module generated by $\{A^k v : k \geq 0,\, v \in \mathbb{Z}^n\}$.

5B) Let M be $\mathbb{C}^3$ with elements considered as column vectors. We make M into a $\mathbb{C}[x]$ module by having x act by left multiplication by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}$$

and by having elements of $\mathbb{C}$ act by scalar multiplication. Find the rank and torsion of this module and give its decomposition as a direct sum of cyclic modules.