January 2000
Algebra Qualifying Exam

1A) Let V be a finite dimensional vector space over a field F, and let U and W be subspaces of V.

(a) Show that $V \neq U \cup W$ unless $V = U$ or $V = W$.

(b) If $\dim(U) = \dim(W)$ show that U and W have a common complement, i.e. there is a subspace X so that $V = U \oplus X = W \oplus X$. (Hint: You may wish to use part (a)).

1B) Let F be a field and V an n-dimensional vector space over F. There is an F-linear endomorphism T of the tensor product $V \otimes V$ mapping $v \otimes w$ to $T(v \otimes w) = w \otimes v$ for all $v, w \in V$. Determine the eigenvalues of T and furthermore determine bases for corresponding eigenspaces.

2A) Suppose G is a group, $H \leq G$, and $x^2 \in H$ for all $x \in G$. Show that $H \lhd G$ and G/H is abelian.

2B) Let G be a finite group, $K \lhd G$ and P a Sylow p-subgroup of K for some prime p. Show that $G = N_G(P)K$.

3A) Suppose R is a principal ideal domain (PID) and $I \neq 0$ is an ideal in R. Show that the set $\{J : J \text{ is an ideal in } R \text{ and } I \subseteq J\}$ is finite.

3B) Let K be a field and let $K[x_1, ..., x_n]$ be the polynomial ring in n indeterminates. Let P be a minimal prime ideal (i.e. a nonzero prime ideal that contains no smaller nonzero prime ideal). Show that P is a principal ideal generated by an irreducible polynomial f.

4A) If $\alpha = \sqrt{3 + \sqrt{15}} \in R$ and set $F = \mathbb{Q}(\alpha)$. Show that F is *not* a Galois extension of $\mathbb{Q}$. Let $K \subseteq \mathbb{C}$ be the Galois closure of F. Determine K explicitly, and in particular find $[K : \mathbb{Q}]$.

4B) Suppose that $f(x) \in \mathbb{Q}[x]$ is irreducible of degree 4. Show that the Galois group of $f(x)$ cannot be the quaternion group $\mathcal{Q}$ of order 8.

5A) Suppose R is a ring with 1, L is a unitary (left) R-module, M and N are submodules of L and both $M + N$ and $M \cap N$ are finitely generated. Show that M and N are finitely generated.

5B) Let T be the $\mathbb{Z}[i]$-module homomorphism from $\mathbb{Z}[i]^2$ to $\mathbb{Z}[i]^2$ defined by the matrix

$$\begin{pmatrix} 2i & 4i+2 \\ 2i-2 & i \end{pmatrix}$$

Determine whether T is one-to-one and whether T is onto.