January 2004
Algebra Qualifying Exam
Solutions

1A) Suppose that A is an $n \times n$ matrix over $\mathbb{R}$ and that $A^{2004} = I$. Show that A^2 is diagonalizable over $\mathbb{C}$.
Answer: Consider the polynomial $f(x) = x^{2004} - 1$. Let $A^2 = B$. Thus $B^{1002} = I$. As B is over $\mathbb{C}$ we know that it has a Jordan canonical form J. Thus there is an invertible matrix P such that $B = PJP^{-1}$. Thus

$$\left(PJP^{-1}\right)^{1002} = PJ^{1002}P^{-1} = I$$

and this is true if and only if J is a diagonal matrix. Thus B is diagonalizable.

1B) Prove or give a counterexample: if V is a vector space over $\mathbb{C}$ and $T : V \longrightarrow V$ is a linear map, then T has an eigenvector.
Answer: Take $V = \mathbb{C}^{\infty}$. Take the linear transformation $(a_1, a_2, ...) \to (0, a_1, a_2, ...)$.

2A) Prove or give a counterexample: if A is an abelian group such that every finitely generated subgroup of A is cyclic, then A is cyclic.
Answer: $\mathbb{Q}/\mathbb{Z}$

2B) Suppose that P is a finite p-group, p a prime, and that P acts on a finite set S, with $p \nmid |S|$.

(a) Show that there exists $s \in S$ such that $gs = s$ for all $g \in P$.

(b) Use (a) to show that $Z(P) \neq 1$ for any nontrivial p-group (where $Z(P)$ denotes the center).

Answer: (a) We know from the orbit stabilizer theorem that $|P| = |Stab_P(s)| |Orbit_P(s)|$. We also know that $|S| = \sum_s |Orbit_P(s)|$, where s is a representative from each orbit. As P is a p-group, we must have that $|P| = p^n$ and so each $|Orbit_P(s)| = p^k$ for some k. Then $|S| = \sum p^{k_i}$ and if each $k_i > 0$ then we have that $|S| = p^{k_1} + \cdots + p^{k_s} = p\left(p^{k_1 - 1} + \cdots + p^{k_s - 1}\right)$ which is not possible as $p \nmid |S|$.

(b) Let $S = P \backslash \{1\}$. Then use part (a).

3A) Suppose that $\varphi : R_1 \to R_2$ is a homomorphism of rings.

(a) If I_2 is an ideal in R_2 show that $\varphi^{-1}(I_2)$ is an ideal in R_1.

(b) Show by example that if I_1 is an ideal in R_1 that it does not necessarily follow that $\varphi(I_1)$ is an ideal in R_2.

Answer: (a) Let $a, b \in \varphi^{-1}(I_2)$. NTS that $a - b \in \varphi^{-1}(I_2)$ and $ra \in \varphi^{-1}(I_2)$ for all $r \in R_1$. As $a, b \in \varphi^{-1}(I_2)$ then we know that $\varphi(a) \in I_2$ and $\varphi(b) \in I_2$. Thus $\varphi(a) - \varphi(b) \in I_2$ as I_2 is an ideal and so $\varphi(a - b) \in I_2$ and so $a - b \in \varphi^{-1}(I_2)$. Now take $a \in \varphi^{-1}(I_2)$. Then consider ra. $\varphi(ra) = \varphi(r)\varphi(a) \in I_2$ as $\varphi(a) \in I_2$ and I_2 is an ideal and thus $ra \in \varphi^{-1} I_2$.

(b) Let $R_1 = \mathbb{Z}$ and $R_2 = \mathbb{Q}$. Then by the injection map φ we can consider the ideal $2\mathbb{Z}$ which is not an ideal in R_2 as the only ideals in a field F are 0 and F.

3B) Define a module M over $\mathbb{R}[x]$ as follows. The underlying abelian group of M is $\mathbb{R}^5$ and the $\mathbb{R}[x]$-module structure is induced by the usual $\mathbb{R}$-vector space structure of $\mathbb{R}^5$ and following action of x :

$$x\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

Determine the elementary divisors and torsion-free rank of M as an $\mathbb{R}[x]$-module.
Answer: Note that the above matrix is in Jordan canonical form. Thus it has invariant factors $(x+1)^2$ and $(x+1)^3$. Thus we know that the module is isomorphic to $\mathbb{R}[x]/(x+1)^2 \oplus \mathbb{R}[x]/(x+1)^3$. The torsion free rank is 0 and the elementary divisors are $(x+1)^2$ and $(x+1)^3$.

4A) Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ does not contain $\mathbb{Q}(\sqrt{5})$.
Answer: Assume that $\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then we can express $\sqrt{5} = a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}$ and thus

$$\begin{aligned} 5 &= \left(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}\right)^2 \\ &= b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6} \end{aligned}$$

where b_i are operatornametions of the a_j. Thus we have that $a_2, a_3, a_4 = 0$ and so $\sqrt{5}$ is rational which is not possible.

4B) Suppose that K is a field, $\mathbb{Q} \leq K \leq \mathbb{C}$, and $[K : \mathbb{Q}]$ is finite. Show that K contains only finitely many roots of unity.
Answer: As $K : \mathbb{Q}$ is finite, there is a finite set of irreducible polynomials that generates the extension field K. As this is a finite set with a finite set of roots, it can only contain a finite set of roots of unity.

5A) If p is a prime, set $A = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})$ and $B = \mathbb{Z}^3$. Determine the elementary divisors and torsion free rank of $A \otimes_{\mathbb{Z}} B$.
Answer: $A \otimes_{\mathbb{Z}} B \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} B \oplus \mathbb{Z}_p \otimes_{\mathbb{Z}} B \cong (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z})^3 \oplus (\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z})^3 \cong (\mathbb{Z}_p)^6$. The elementary divisors are $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$. The torsion free rank is 0.

5B) Prove or give a counterexample: If $f : A \longrightarrow B$ is an injective homomorphism of $\mathbb{Z}$-modules, and C is a $\mathbb{Z}$-module, then the map

$$\begin{aligned} f \otimes 1 &: A \otimes_{\mathbb{Z}} C \longrightarrow B \otimes_{\mathbb{Z}} C \\ a \otimes c &\longrightarrow f(a) \otimes c \end{aligned}$$

is injective.
Answer: Counter example: Consider $A = \mathbb{Z}$, $B = 2\mathbb{Z}$, and $C = \mathbb{Z}_2$. Take the map f as $f(a) = 2a$. Then $f(a) \otimes c = 2a \otimes c = 0$ for all a and thus has a nontrivial kernel.