1 Module Test - Review

Selected Solutions

1. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$-modules.

   Answer: Define a module homomorphism $f : \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \to \mathbb{Q}$ by $f\left(\frac{a}{b} \otimes \frac{c}{d}\right) = \frac{ac}{bd}$ is a well-balanced homomorphism. Check! We also know that f has an inverse g,

   $$g : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \text{ by } f(a/b) = \frac{a}{b} \otimes 1.$$

   Check!

2. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

   Answer: Free if and only if it has a basis. Assume $\mathbb{Q}$ has a basis. It must have size bigger than 1. As if we assume a/b is the only element and p is a prime not dividing b, then $1/p$ cannot be expressed as $z \cdot \frac{a}{b} = \frac{1}{p}$. So a basis must have two elements. But any two elements, a/b and c/d are linearly dependent as $(cb)(a/b) - (ad)(c/d) = ac - ac = 0$.

3. Describe the abelian group with the presentation

   $$A = \langle a, b, c : 4a + 10b - 8c = 0, 2a + 8b - 4c = 0\rangle$$

   Answer: $\begin{pmatrix} 4 & 10 & -8 \\ 2 & 8 & -4 \end{pmatrix} \to \begin{pmatrix} 2 & 8 & -4 \\ 4 & 10 & -8 \end{pmatrix} \to \begin{pmatrix} 2 & 8 & -4 \\ 0 & -6 & 0 \end{pmatrix} \to$

   $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \end{pmatrix}$, thus $A \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

4. Let F be a field and V and n dimensional vector space over F. There is an F-linear endomorphism T of the tensor product $V \otimes V$ mapping $v \otimes w$ to $T(v \otimes w) = w \otimes v$ for all $v, w \in V$. Determine the eigenvalues of T and further determine bases for corresponding eigenspaces.

   Answer: Take $\{v_1, ..., v_n\}$ as a basis for V. So $V\otimes V$ has a basis $\{v_i \otimes v_j : 1 \leq i, j, \leq n\}$.
   Order the basis as $\left\{ \begin{array}{c} v_1 \otimes v_1, ..., v_n \otimes v_n, v_1 \otimes v_2, v_2 \otimes v_1, ..., v_1 \otimes v_n, \\ v_n \otimes v_1, v_2 \otimes v_3, v_3 \otimes v_2, ..., v_{n-1} \otimes v_n, v_n \otimes v_{n-1} \end{array} \right\}$.

   Then T can be given as the matrix

   $$\begin{pmatrix} I_n & & & & & & \\ & 0 & 1 & & & & \\ & 1 & 0 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}. \; Det(T - xI) = (1-x)^n \left(x^2 - 1\right)^{(n^2-n)/2}$$

So we have eigenvalues of 1 with multiplicity $n+\left(n^2-n\right)/2$ and -1 with multiplicity of $\left(n^2-n\right)/2$. The eigenspace corresponding to 1 has basis

$$\left\{\begin{array}{l} v_i\otimes v_i : 1\leq i\leq n \\ v_i\otimes v_j + v_j\otimes v_i : 1\leq i<j\leq n \end{array}\right\}$$

and the eigenspace corresponding to -1 has basis

$$\{v_i\otimes v_j - v_j\otimes v_i : 1\leq i<j\leq n\}.$$

5. Proof or counterexample:

   (a) If R is a PID and M is a finitely generatored torsion-free R-module then M is free.

   Proof: If R is a PID and M is finitely generated torsion-free R-module, then M is free. $M\cong R/I_1\oplus ...\oplus R/I_n$ where $I_1\subseteq ...\subseteq I_n$. Since M is torsion free each ideal I_k is the zero ideal and thus $M\cong R^n$ is free.

   (b) If R is an ID and M is a finitely generated torsion free R-module, then M is free.

   False: Take $R=\mathbb{Z}[x]$ and $M=\langle 2,x\rangle$ as a module over R.

   (c) Every submodule of a free module is free.

   False: Take $\mathbb{Z}_4$ as a $\mathbb{Z}_4$ module. It is free as it has a basis of $\{1\}$. Take $M=2\mathbb{Z}_4$ as a $\mathbb{Z}_4$-module. $2\mathbb{Z}_4=\{2,0\}$ and there is no linearly independent set to take. (We also needed PID).

   (d) R is commutative with 1; M an R-module implies that M is a finite set if and only if finitely generated and every element is a torsion element.

   False: Take $\mathbb{Z}\oplus\mathbb{Z}$ as a ring (not an ID). Take $M=0\oplus\mathbb{Z}$ as a $\mathbb{Z}\oplus\mathbb{Z}$ module. Finitely generated: $(0,1)$. All elements are torsion. Not a finite set.

   (e) If E and F are free R-modules, then $E\oplus F$ is free.

   Answer: TRUE

6. Find the characteristic polynomial, minimal polynomial, rational cannonical form, and the JCF of $A=\begin{pmatrix}0&4&0\\2&0&8\\0&-1&0\end{pmatrix}$.

   Answer: $Charpoly=-x^3$, $m_A(x)=-x^3$, $RCF=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}=JCF.$

7. Suppose A and B are finitely generated abelian groups. View A and B as $\mathbb{Z}$-modules. Compute $A\otimes_{\mathbb{Z}}B$ as explicitly as possible.

8. Let $0\to A\to B\to C\to 0$ be an exact sequence of R-modules where R is any ring with 1. Prove that if B has torsion elements then either A or C has torsion elements.

9. Let M be a unitary cyclic R-module, R a ring with 1. Show that $M \cong R/I$ for some left ideal I in R.

   Answer: Define a map $f : R \to M$ by $f(r) = rm$, $r \in R$ and m a fixed generator of M as M is cyclic. As M is cyclic, the map is onto. Check that it is a homomorphism! Then kernel of the map if $A(m)$, the annihilator of m which forms an ideal. Check! Thus by the fundamental homomorphism theorem we have $R/A(m) \cong M$.

10. Let M be an R-module and let A, B, C be submodules. If $C \subseteq A$, prove that

$$A \cap (B + C) = (A \cap B) + C.$$

11. Suppose that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\phi} & M & \xrightarrow{\phi'} & M_2 & \longrightarrow & 0 \\ & & \downarrow_f & & \downarrow_g & & \downarrow_h & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{\psi} & N & \xrightarrow{\psi'} & N_2 & \longrightarrow & 0 \end{array}$$

   is a commutative diagram of R-modules and R-module homomorphisms. Assume that the rows are exact and that f and h are isomorphisms. Prove that g is an isomorphism.

12. Suppose F is a field, A and B are $n \times n$ matrices over F and $A' = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ is similar to $B' = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Show that A and B are similar over F.

13. Use Smith Normal Form to find all integral solutions of the equation

$$\begin{aligned} 2x_1 - 7x_2 + 12x_3 &= 4 \\ -4x_1 + 3x_2 - 2x_3 &= -8 \end{aligned}$$

14. Suppose $T : V \to V$ is a linear transformation on a finite dimensional vector space V over a field F, and that T has invariant factors $x-1, x(x-1)$, and $x(x-1)^2$.

   (a) What is $\dim_F V$?

   (b) Is T one-to-one?

   (c) What is the minimal polynomial of T.

   (d) What are the RCF and JCF?

15. An $n \times n$ matrix A over a field F is called nilpotent if $A^k = 0$ for some k.

   (a) Is A diagonalizable?

   (b) Does A necessarily have a JCF? If so what does it look like?

16. An R-module P is called projective if given any modules M and N with $M \twoheadrightarrow N$ and $f : P \to N$, then there exists a F s.t. the following diagram commutes.

$$\begin{pmatrix} & & P \\ & {}^{F} \swarrow & \downarrow_f \\ M & \overset{\phi}{\twoheadrightarrow} & N \end{pmatrix}, \text{ i.e. } \phi F = f.$$

Prove that this implies that P is a direct summand of a free module.

17. Suppose R is a ring with 1, L is a unitary R-module, M and N are submodules of L and both $M + N$ and $M \cap N$ are finitely generated. Show that M and N are finitely generated.