Math 511a - Test 2 Practice
Practice Questions for Rings

1. Suppose R is a UFD and $r \in R^*$. Show that there are only finitely many $s \in R$ such that $(s) \geq (r)$.

2. If F is a field and $f(x), g(x) \in F[x]$, whos that the least common multiple $lcm(f(x), g(x))$ is a generator for the ideal $(f(x)) \cap (g(x))$.

3. Let $R = M_2(\mathbb{Z})$, the ring of 2×2 matrices over $\mathbb{Z}$, and $M = M_2(2\mathbb{Z})$. Show that M is a maximal ideal in R, and that $R/M \cong M_2(\mathbb{Z}_2)$.

4. Suppose R and S are nontrivial rings with 1 and $\phi : R \to S$ is a homomorphism such that $\phi(1_R) \neq 0$. IF $\phi(1_R) \neq 1_S$, show that $\phi(1_R)$ is a zero-divisor in S. Conclude that if S is an integral domain, then $\phi(1_R) = 1_S$.

5. Suppose R is a finite commutative ring with 1. Show that every prime ideal of R is maximal.

6. Suppose that $S = \mathbb{Q}[x]$ as an additive group, but that the usual multiplication of polynomials is replaced by composition, i.e. $(f \circ g)(x) = f(g(x))$. Show that S is not a ring.

7. If I is an ideal in a ring R, show that $A(I) = \{r \in R : rI = 0\}$ is an indeal in R.

8. If R is a commutative ring with 1 and I is an ideal in R define
   $\sqrt{I} = \left\{r \in R : r^k \in I,\ \text{some } k \in \mathbb{N}\right\}$.

   (a) Show that $\sqrt{I}$ is an ideal and that $I \subseteq \sqrt{I}$.

   (b) If P is a prime ideal in R with $I \subseteq P$ show that $\sqrt{I} \subseteq P$.

   (c) If $R = \mathbb{Z}$ and $I = (72)$ calculate $\sqrt{I}$.

9. True or False (proof or counterexample)

   (a) If F and K are fields, R is a ring, and $F \subseteq R \subseteq K$, then R is a field.

   (b) If R is a Euclidean domain and S is a nonzero subring then S is Euclidean.

   (c) There is a ring R with 10 elements such that if $a, b \in R$, $a \neq 0$, $b \neq 0$, then $ab \neq 0$.

   (d) Every UFD is a PID.

   (e) If $r \in R$ (commutative ring with 1) then the set $I = \{r \in R : \exists x \neq 0 = rx\}$ is an ideal.

(f) If $r \in R$ (comm w/1) then $r = \pm 1$.

(g) If J is a prime ideal then R/J is a field.

(h) $\mathbb{C}[x] / \left(f(x) = x^3 - x\right) \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$

10. Let R be a PID. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an increasing sequence of ideals in R. Prove that the sequence is everntually constant, i.e. for some n, $I_n = I_{n+1} = I_{n+2} = \cdots$

11. Give an example of an irreducible polynomial of degree 100 in $\mathbb{C}[x, y]$.

(a) Give an example of an ideal in a commutative ring which is prime but not maximal.

(b) Prove that if $f : R \to S$ is a homomorphism of commutative rings and $I \subseteq S$ is a prime ideal, then $f^{-1}(I)$ is prime.

12. Show that every nonzero prime ideal in the ring $Z[i]$ of Gaussian integers is maximal.

13. Suppose R is an ID (with 1) having only finitely many ideals. Prove that R is a field. What if R is just a commutative ring, not a domain?

14. Give an example of a prime ideal in $\mathbb{C}[x, y]$. Find one that is prime but not maximal. Give one that is not principal.

15. Let R be a commutative ring. Recall that $r \in R$ is nilpotent if $r^n = 0$ for some $n > 0$ and that the set of all nilpotent elements in R is an ideal. Show that R/N has no nonzero nilpotent elements.

16. Show that the field $\mathbb{C}$ of complex numbers is isomorphic with the subring

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

of $M_2(\mathbb{R})$.

17. Let R be the addivite abelian $\mathbb{Z} \oplus \mathbb{Z}$ and let R' be the ring $\operatorname{End}(A)$, with multiplication being composition of functions. Show by example that R' is not commutative.

18. Assume that $\gcd(m, n) = 1$. Prove that $\mathbb{Z}/(mn)\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$.

19. Let R be a commutative ring, and let $M \subseteq R$ be an ideal. Prove that R is a local ring with maximal ideal M if and only if every element of R not in M is invertible. (Recall that a ring is local if it has a unique maximal ideal).