PROFESSIONAL DEVELOPMENT WORKSHOP · FALL 2005
ASSIGNMENT 6

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The Rule of Four, with emphasis on writing

Describe the Rule of Four

The Rule of Four is that concepts should be presented graphically, algebraically, numerically, and verbally. Some writers replace the last with "in context with applications", and in fact this is the approach I take. (See also my section "Writing problems" below.)

There are many graphical problems, and in fact I think these are easiest for students — we are after all visual, pattern-recognizing creatures, with a large fraction of our brains devoted to visual processing. Examples are asking whether a relation is a function by the vertical-line test, whether a function is one-to-one by the horizontal-line test, etc.

Algebraic problems are my favorite, if I were left to my own devices — this is what I feel math is centrally about. Examples are finding the vertex of a parabola by completing the square, computing x-intercepts and y-intercepts by setting output or input equal to zero, respectively, and using properties of exponents to obtain properties of the log function.

Numerical problems include, I would contend, using the graphing calculator to estimate a zero of a function. Also I would place into this category any problems involving the use of a table of (x, y) pairs, e.g. when we start with an equation $y = f(x)$ and ask students to tabulate some points on the graph.

"In context with applications" refers to any word problem. One example is finding the doubling time for an investment. Another example is thinking of a horizontal asymptote in terms of the long-term behavior of a physical situation (e.g. cooling of a hot object to the ambient temperature).

Positives and negatives

I believe the Rule of Four is vital for my understanding as a graduate student — professors when left to their own devices will often only do things algebraically, and it is up to me as a graduate student to come up with my own graphs, numerical experiments and applications. Since I like it so well, I of course want to pass this on to my students, and will continue doing so. I believe that the Rule of Four is vital for intuition-building. As much as we might think that math (at any level) rests on the recall of definitions and theorems, I believe that our actual mastery depends centrally on our intuition, with rigor applied like a coat of paint.

The question is, does it *work* for everyone? I'm not convinced students always comprehend the relationships. Presenting things in four ways might, at some times and for some students, actually be more confusing than presenting them in a single way. Is a logarithm an exponent, or an inverse function? Can it really simultaneously be both in a freshman's head?

One particular example is the following: We sometimes present a function which is defined *solely* by the (x, y) pairs in a table, e.g.

x	1	2	3	4
y	5	7	9	2

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The domain of this function has only four numbers in it. Yet I've found that several students, when confronted with such a table, feel the urge to fit some kind of function to it and (whether they realize it or not) extend the domain to all real numbers. Sometimes, that's what we want — e.g. taking some data and fitting a linear function to it is one of the most useful, real-world skills I can teach to my students. Yet when the situation doesn't call for that, some students fail to detach.

There is a skill which I as a graduate student am still trying to acquire. One type of understanding, which I already possess, is *analytic*: given some information, we systematically follow a process to obtain a result — be it the quadratic formula, the Gram-Schmidt orthonormalization procedure, or Buchberger's algorithm. These skills may be easy or difficult, but once mastered, *there is no doubt what to do next.* The other kind of understanding is *synthetic*: given some information, we must free-associate for possible solutions. Qualifier questions are all about this. Rarely will a qual ask, "Given this basis, find an orthonormal basis." These questions are much harder because they depend on the *simultaneous, parallel* recall of many, many facts. There is *no* clear indication of what to do next — we must find our own way. To a learner, this smells like black magic: One feels stumped; one is given a solution which after the fact seems obvious, but then one asks, "How would I have *known* to apply the nilpotence property in this context?"

I find myself propagating the same two kinds of understanding to my students. Sometimes I ask my students to use the quadratic formula; they know it and they apply it. After the first exam, explaining a question which most students missed, I found myself saying, "Well, all you need to do now is to recall that all numbers have a unique cube root . . . ." Students feel frustrated by this — yes, they could take the cube root of any number if specifically asked, but how could they have known to pull that particular rabbit out of the hat at that particular time? My point is that the Rule of Four can turn into this — given, say, an equation, if we say, "Well, all you need to do know is graph it," students may feel perplexed. The good news is that there are only four things in the Rule of Four, so there are only four hats. So, this can be turned into a procedure: for example, *always* try drawing a picture, whenever you're stumped.

Writing problems

I have spent significant class time on word problems — focusing on the kinds of problem-solving skills I want my students to have as they go forward with their educational and professional careers. But those are really *reading* problems, if you will — they involve words, but the student is the consumer of those words rather than the producer. I have spent little time on writing problems. Ruud and Shell have, at the end of most sections, a few open-ended questions which require a written response. I would guess that less than one in ten of the problems I assign are of this form. I do not grade these problems closely. For example, $\log_{10}(1000)$ is or is not 3; I can mark a student's answer right or wrong. On the other hand, when Ruud and Shell ask why a graphing calculator's rendition of $y = \sqrt{1 - x^2}$ doesn't touch the x axis — there is no clear answer. I *do* want to ask these questions, to get the students thinking, and to find out what they have to say. I do *not* want to grade these questions in a right-or-wrong way; for me, that defeats the purpose. I assign some homework every day – somewhere between 5-15 problems. Of those, I grade 5 problems on each assignment. On assignments which include writing problems, I don't grade the writing problems for credit, but I do read them and provide some brief written feedback.

As well, I know that some students will go ahead and take brief calculus, statistics, maybe business math. But for most of them, this is a terminal course. I openly acknowledge that my goal is to help them prepare for the final — which is multiple-choice as well as difficult. So, I don't think writing problems are a priority.

Student response

My students' response to the Rule of Four: In general, when I switch from the algebraic to the graphical point of view or vice versa, a common response is surprise — "Wow, I hadn't thought of it that way." I have no further comment — maybe I should solicit some student feedback about the Rule of Four.

My students respond to writing problems in three main ways: (1) Some provide the briefest of answers, as if getting a nuisance out of the way. (2) Some go on for quite a while, with some formal-sounding writing that really doesn't make sense. I think they feel like they're supposed to say something insightful, but don't really know how. (3) Some really think about the problem and come to interesting conclusions, or at least point out what it is that they don't understand. I prefer answers of the third form. I'd not thought much about writing problems until this 597T assignment. Up until now, I'd not asked for a certain type of answer — I left it wide open, intentionally trying to find out what these students would say. In the future, I will do more to actively encourage responses of the third form.

Comment on the reading

Rishel's section on writing assignments is a nice little note. He addresses reasonable concerns about writing assignments: they take a lot of time to write and to grade; math instructors may not have good writing skills themselves; writing may seem irrelevant in math. He successfully counters these arguments: he provides references for the how-to's of creating writing assignments; he suggests grading for mathematical content rather than grammar; he suggests that writing is actually of vital importance. The only thing I didn't like about this section was that it was too short. For example, he might have taken a few pages to summarize some of the techniques presented in the sources he cites. Those few additional pages would have been well-spent.

I most like Rishel's contention that writing forces students into active recall and understanding of a subject. For example, we all can (and do) learn to mechanically solve certain types of equations. However, if a student writes a summary of the day's lecture for a hypothetical fellow student who has missed class, that student will be led to think about *why* the techniques work, what order to present them in, etc. — concepts which a computational, drill-oriented homework assignment would not bring out.