# Exam #4 solutions · Thursday, April 26, 2007

MATH 124 · Calculus I · Section 8 · Spring 2007

*Note: Some of my solutions are wordy, for the sake of explanation. All I expect of you is computation, unless a problem specifically requests a verbal response.*

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**Problem 1.** Let

$$\begin{aligned} x(t) &= 3t^2 - 6t \\ y(t) &= \frac{4}{3}t^3 - 4t. \end{aligned}$$

**Part (a).** Find the time(s) $t$, if any, when the particle comes to a stop.

*Solution:* We need to find when $dx/dt$ and $dy/dt$ are both zero. We have

$$\begin{aligned} dx/dt &= 6t - 6 \\ dy/dt &= 4t^2 - 4. \end{aligned}$$

The former is zero when $t = 1$; the latter is zero when $t = \pm 1$. They are simultaneously zero only at $t = 1$.

**Part (b).** Find an equation for the tangent line to this curve at $t = 3$.

*Solution:* Parametric equations for a line through a point $(x_0, y_0)$ in the direction $a, b$ are

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt. \end{aligned}$$

The point $(x_0, y_0)$ is found by plugging $t = 3$ into the original equations:

$$\begin{aligned} x_0 &= 3 \cdot 3^2 - 6 \cdot 3 = 9 \\ y_0 &= \frac{4}{3}3^3 - 4 \cdot 3 = 24. \end{aligned}$$

The $a$ and $b$ are $dx/dt|_{t=3}$ and $dy/dt|_{t=3}$:

$$\begin{aligned} a &= \left.\frac{dx}{dt}\right|_{t=3} = 6 \cdot 3 - 6 = 12 \\ b &= \left.\frac{dy}{dt}\right|_{t=3} = 4 \cdot 3^2 - 4 = 32 \end{aligned}$$

So, the tangent line has equations

$$\begin{aligned} x &= 9 + 12t \\ y &= 24 + 32t. \end{aligned}$$

**Problem 2.** The function $H(t)$ describes the growth rate in thousands per month of flour beetles in a jar, where $t$ is measured in months since the start of the year.

**Part (a).** What are the units of $\int_4^7 H(t)dt$?

*Solution:* The units of an integral are the units of the integrand times the units of the dependent variable. Since $H(t)$ is thousands of beetles per month and $t$ is months, the units of the integral are thousands of beetles.

**Part (b).** Give a practical interpretation of $\int_4^7 H(t)dt$.

*Solution:* This is the population change between month 4 and month 7.

**Problem 3.** The air pressure within a chamber is given by

$$P(t) = 2.1 + 0.4t^{0.5}$$

where $P$ is in units called atmospheres and $t$ is measured in hours. Find the average pressure over the time between $t = 2$ hours and $t = 5$ hours.

*Solution:* Compute

$$\frac{\int_2^5 P(t)}{5-2} = \frac{\int_2^5 (2.1 + 0.4t^{0.5})}{5-2}.$$

The integral is

$$2.1t + \frac{0.4t^{1.5}}{1.5}\Big|_2^5 = \left(10.5 + \frac{0.4 \cdot 5^{1.5}}{1.5}\right) - \left(4.2 + \frac{0.4 \cdot 2^{1.5}}{1.5}\right) \approx 8.53.$$

Then the average value is

$$8.53/3 \approx 2.84.$$

Note that if you graph $P(t)$ from 2 to 5, you'll see that it rises slowly from $P(2) \approx 2.7$ to $P(5) \approx 3.0$. Thus, an average of 2.84 is reasonable.

**Problem 4.** Find the exact area between $f(x) = e^x - 2$ and $g(x) = -1$ on the interval $[2, 4]$.

*Solution:* The area between $f(x)$ and $g(x)$ from 2 to 4 is $\int_2^4 (f(x) - g(x))\, dx$ so we can compute

$$\begin{aligned} \int_2^4 (e^x - 2 - (-1))\, dx &= \int_2^4 (e^x - 1)\, dx \\ &= e^x - x\Big|_2^4 \\ &= (e^4 - 4) - (e^2 - 2) \\ &= e^4 - e^2 - 2. \end{aligned}$$

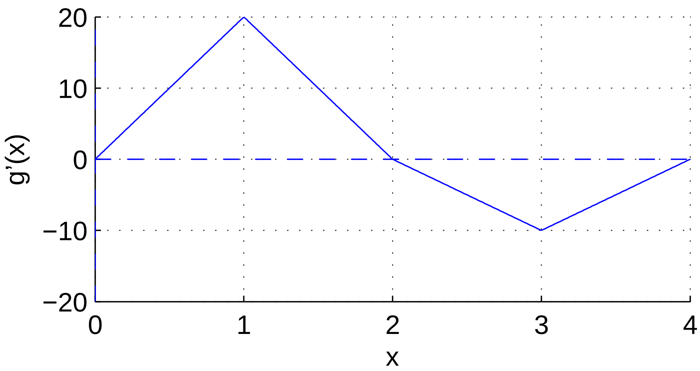
**Problem 5.** Find the general antiderivative:

$$\int \left(\frac{y^{2.1}}{3} - \frac{7}{y} + 0.2Ae^y + B\right)\, dy.$$

*Solution:*

$$\frac{y^{3.1}}{9.3} - 7\ln(|y|) + 0.2Ae^y + By + C.$$

**Problem 6.** Let $g'(x)$ be given by the following graph, and suppose $g(0) = 2$:




**Part (a).** What are the $x$-coordinates of the critical points of $g(x)$?

*Solution:* Critical points of $g(x)$ occur when $g'(x)$ is zero (or undefined). From the graph we see $g'(x) = 0$ at $x = 0, 2, 4$.

**Part (b).** What are the $x$-coordinates of the inflection points of $g(x)$?

*Solution:* Inflection points of $g(x)$ occur at extrema of $g'(x)$. These are at $x = 1, 3$.

**Part (c).** Find the values of $g(x)$ at the critical and inflection points.

*Solution:* We are given $g(0) = 2$, and we are given $g'(x)$ in graphical format. This suggests the fundamental theorem of calculus, one landmark at a time. By "landmark" I mean a critical or inflection point.

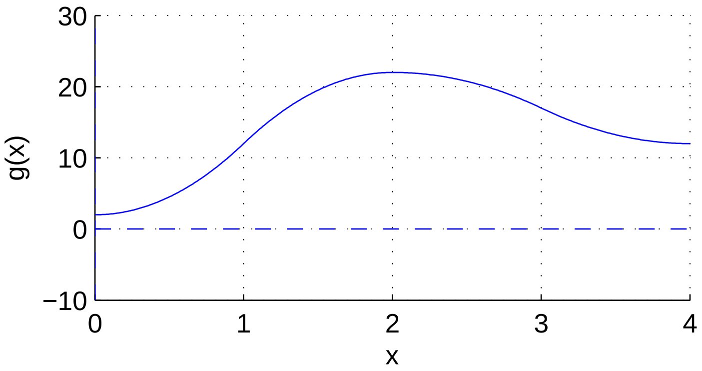
$$\begin{aligned} g(1) &= g(0) + \int_0^1 g'(x)\,dx = 2 + 10 = 12. \\ g(2) &= g(1) + \int_2^1 g'(x)\,dx = 12 + 10 = 22. \\ g(3) &= g(2) + \int_3^2 g'(x)\,dx = 22 - 5 = 17. \\ g(4) &= g(3) + \int_4^3 g'(x)\,dx = 17 - 5 = 12. \end{aligned}$$

**Part (d).** Sketch a graph of $g(x)$. Label critical points and inflection points of $g(x)$.
*Solution:* Using the above information we can plot the critical points $(0, 2)$, $(2, 22)$, and $(4, 12)$, along with the inflection points $(1, 12)$ and $(3, 17)$. Then we can connect the dots as follows:

- Since $g'(x)$ is positive on $(0, 2)$, $g(x)$ is increasing there.
- Since $g'(x)$ is negative on $(2, 4)$, $g(x)$ is decreasing there.
- Since $g'(x)$ increases on $(0, 1)$ and $(3, 4)$, $g(x)$ is concave up there.
- Since $g'(x)$ decreases on $(1, 2)$ and $(2, 3)$, $g(x)$ is concave down there.

(The sketch is on the next page.)

Here is a sketch:




**Problem 7.** The quantity $A$ varies with time as specified by

$$\frac{dA}{dt} = 7.3\cos(t) - 0.04.$$

**Part (a).** Write down a general solution for $A$.

*Solution:* The general antiderivative is

$$7.3\sin(t) - 0.04t + C.$$

**Part (b).** Given that $A(0) = 2.1$, write down a specific solution for $A$.

*Solution:* Using the initial condition, we have

$$2.1 = 7.3\sin(0) - 0.04 \cdot 0 + C = C.$$

So, the solution is

$$7.3\sin(t) - 0.04t + 2.1.$$

**Problem 8.**
**Part (a).** Let

$$G(x) = \int_1^x e^{t^2}\,dt.$$

Determine (with justification) whether $G(x)$ is increasing, decreasing, or constant.

*Solution:* A function is increasing, decreasing, or constant when its derivative is positive, negative, or zero, respectively. The derivative is

$$G'(x) = \frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}$$

using the second fundamental theorem of calculus. The output of exponential functions is always positive, so this function is increasing everywhere.
Here is another solution, which a student came up with: $e^{t^2}$ is always positive, so as the right endpoint of the integral slides rightward from 1, only positive areas are added. Thus, the integral function is increasing.

**Part (b).** Now let

$$G(x) = \int_1^{ax+b} e^{t^2}\,dt.$$

Find $G'(x)$.

*Solution:* Using the second fundamental theorem and the chain rule, we evaluate the integrand at the right endpoint (upper limit), times the derivative of the right endpoint. This is

$$G'(x) = ae^{(ax+b)^2}.$$