Exam #4 solutions · Fri. Dec. 2, 2005

MATH 110 · Section 10 · Fall 2005 Name ______________________________

REMINDER: *The final exam is Monday, December 12, 8:00 a.m. to 10:00 a.m., in CESL 103. Please arrive at 7:40 a.m.*

Formulas:

$A = Pe^{rt}$	$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
$a_n = a_1 + (n-1)d$	$S_n = n\left(\frac{a_1+a_n}{2}\right)$	$S_n = \frac{n}{2}(2a_1 + (n-1)d)$
$a_n = a_1 r^{n-1}$	$S_n = a_1\left(\frac{1-r^n}{1-r}\right), r \neq 1$	$\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1-r}, \lvert r\rvert < 1$

Problem 1. (**6 points**) One may approximate the altitude A (in meters above sea level) using the formula

$$A = 22,860 \ln\left(\frac{p_0}{p}\right)$$

where p is the atmospheric pressure in mm Hg at that altitude, and p_0 is the sea-level atmospheric pressure, 760 mm Hg. At an altitude of 4,000 meters, what is the approximate atmospheric pressure?

Answer. We are given $p_0 = 760$ and $A = 4000$, and we are asked to find p:

$$\begin{aligned}
4000 &= 22860 \ln\left(\frac{760}{p}\right) \\
\frac{4000}{22860} &= \ln\left(\frac{760}{p}\right) \\
e^{4000/22860} &= \frac{760}{p} \\
p &= \frac{760}{e^{4000/22860}} \approx 638.
\end{aligned}$$

Problem 2. (**6 points**) Your aunt Judy gave you a generous high-school graduation gift of \$4,000, which you wisely invested in an account yielding 6% annual interest (compounded continuously). How long (in years, to the nearest 0.1 year) will it take for the balance to reach \$10,000?

Answer. This is a compound-interest problem, so use the compound-interest formula, $A = Pe^{rt}$. We are given $P = 4000$, $A = 10000$, and $r = 0.06$, and we are asked to find t:

$$\begin{aligned}
10000 &= 4000e^{0.06t} \\
\frac{10000}{4000} &= e^{0.06t} \\
2.5 &= e^{0.06t} \\
\ln(2.5) &= 0.06t \\
t &= \frac{\ln(2.5)}{0.06} \approx 15.3.
\end{aligned}$$

Problem 3. (**6 points**) Consider the sequence whose terms given by

$$a_n = \frac{(-1)^n}{n^2+1}.$$

Which of the following statements are true about the 6th and 7th terms of the sequence?

$$(1) a_6 < 0 \qquad (2) a_7 < 0 \qquad (3) a_6 < a_7$$

(A) 1, 2, and 3 (B) 1 and 2 only (C) 1 only (D) 2 only (E) None of these

Answer. Given the formula

$$a_n = \frac{(-1)^n}{n^2+1},$$

we can compute

$$a_6 = \frac{(-1)^6}{6^2+1} = \frac{1}{37} \quad \text{and} \quad a_7 = \frac{(-1)^7}{7^2+1} = \frac{-1}{50}.$$

Since $a_6 > 0$ and $a_7 < 0$, only (2) applies.

Problem 4. Consider the sequence

$$\frac{3}{2}, \frac{4}{4}, \frac{5}{6}, \frac{6}{8}, \frac{7}{10}, \ldots.$$

(a) (**4 points**) Find a closed-form expression for a_n. (This means that the formula for a_n should not refer to a_{n-1}.)

Answer. Write down the table of values

n	a_n
1	3/2
2	4/4
3	5/6
4	6/8
5	7/10

The numerators are each 2 more than n, and the denominators are each twice n. So the nth term is

$$a_n = \frac{n+2}{2n}.$$

(b) (**4 points**) What is the 20th term of the sequence?

Answer. Evaluate the formula from part (a) at $n = 20$:

$$a_{20} = \frac{20+2}{2 \cdot 20} = \frac{22}{40} = \frac{11}{20} = 0.55.$$

(c) (**4 points**) Describe the long-term behavior of the sequence: does it approach a specific value? If so, what and why? If not, why not?

Answer. The sequence is the rational function $\frac{x+2}{2x}$ for positive integer inputs, which has horizontal asymptote 1/2. This can be checked by computing a few more terms of the sequence.

Problem 5. (**5 points**) Which of the following sequences are arithmetic?

$$(1) 1, -1, 1, -1, 1, \ldots \qquad (2) 2, 2, 2, 2, 2, \ldots \qquad (3) 1, 4, 9, 16, 25, \ldots$$

(A) 2 only (B) 3 only (C) 2 and 3 only (D) 1 and 2 only (E) None of these

Answer. A sequence is arithmetic if its adjacent differences are all the same.

- Sequence 1 is not arithmetic since its differences alternate between -2 and 2.
- Sequence 2 is arithmetic since its differences are all 0.
- Sequence 3 not is arithmetic since its differences are not the same: 3, 5, 7, ...

Problem 6. (**5 points**) Which of the following sequences are geometric?

$$(1) 1, -1, 1, -1, 1, \ldots \qquad (2) 2, 2, 2, 2, 2, \ldots \qquad (3) 1, 4, 9, 16, 25, \ldots$$

(A) 2 only (B) 3 only (C) 2 and 3 only (D) 1 and 2 only (E) None of these

Answer. A sequence is geometric if its adjacent ratios are all the same.

- Sequence 1 is geometric since its adjacent ratios are all -1.
- Sequence 2 is geometric since its adjacent ratios are all 1.
- Sequence 3 is not geometric since its ratios are not the same: 4, 9/4, 16/9, ...

Problem 7. (**6 points**) Given that

$$\sum_{k=1}^{10} ck = 33,$$

determine the value of c.

Answer. Use the distributive property to obtain

$$\sum_{k=1}^{10} ck = c \sum_{k=1}^{10} k.$$

Then, use the formula from page 1 to obtain

$$\sum_{k=1}^{10} k = \frac{10 \cdot 11}{2} = 55.$$

Then

$$c \cdot 55 = 33$$

and so

$$c = 3/5.$$

Problem 8. (**6 points**) Evaluate the sum

$$\sum_{k=4}^{20} 7k^2.$$

Answer. Use the distributive property to get

$$\sum_{k=4}^{20} 7k^2 = 7\sum_{k=4}^{20} k^2.$$

To get the sum from 4 to 20, take the sum from 1 to 20, and subtract the sum from 1 to 3. Use the formula from page 1 to obtain values for these:

$$\begin{aligned}
\sum_{k=4}^{20} 7k^2 &= 7\sum_{k=4}^{20} k^2 \\
&= 7\left(\sum_{k=1}^{20} k^2 - \sum_{k=1}^{3} k^2\right) \\
&= 7\left(\frac{20\cdot 21\cdot 41}{6} - \frac{3\cdot 4\cdot 7}{6}\right) \\
&= 7\,(2870 - 14) \\
&= 7\,(2856) \\
&= 19992.
\end{aligned}$$

Problem 9. (**6 points**) Evaluate the sum

$$\sum_{k=1}^{8} 4\left(\frac{1}{3}\right)^{k-1}.$$

Answer. This is a finite geometric sum with $a_1 = 4$ and $r = 1/3$, so use the formula for finite geometric sums from page 1:

$$\begin{aligned}
\sum_{k=1}^{8} 4\left(\frac{1}{3}\right)^{k-1} &= \sum_{k=1}^{8} a_1 r^{k-1} \\
&= a_1\left(\frac{1-r^8}{1-r}\right) \\
&= 4\left(\frac{1-\left(\frac{1}{3}\right)^8}{1-\frac{1}{3}}\right) \\
&\approx 5.999.
\end{aligned}$$

Problem 10. (**6 points**) Evaluate the sum

$$\sum_{k=1}^{\infty} 4\left(\frac{1}{3}\right)^{k-1}.$$

Answer. This is an infinite geometric sum with $a_1 = 4$ and $r = 1/3$, so use the formula for infinite geometric sums from page 1:

$$\begin{aligned}\sum_{k=1}^{\infty} 4\left(\frac{1}{3}\right)^{k-1} &= \sum_{k=1}^{\infty} a_1 r^{k-1} \\ &= \frac{a_1}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4}{\frac{2}{3}} = \frac{4\cdot 3}{2} = 6.\end{aligned}$$

Problem 11. (**6 points**) Find a closed-form expression for the nth term of the arithmetic sequence with first term 7.2 and common difference 0.1.

Answer. This is an arithmetic sequence with $a_1 = 7.2$ and $d = 0.1$. So, use the formula for arithmetic sequences:

$$\begin{aligned}a_n &= a_1 + (n-1)d \\ &= 7.2 + (n-1)0.1 \\ &= 7.2 + 0.1n - 0.1 \\ &= 7.1 + 0.1n.\end{aligned}$$

Problem 12. (**6 points**) Find the number of terms in the sequence

$$11.3, 11.5, 11.7, \ldots, 57.5.$$

Answer. This is an arithmetic sequence with $a_1 = 7.2$, $a_n = 0.1$, and $d = 0.1$. So, use the formula for arithmetic sequences:

$$\begin{aligned}a_n &= a_1 + (n-1)d \\ 57.5 &= 11.3 + (n-1)0.2 \\ 46.2 &= (n-1)0.2 \\ \frac{46.2}{0.2} &= n-1 \\ n-1 &= 231 \\ n &= 232.\end{aligned}$$

Problem 13. (**6 points**) Find the sum of the even positive integers up to and including 500.

Answer. This is an arithmetic sum with $a_1 = 2$, $a_n = 500$, and $d = 2$. The formula for arithmetic sums requires us to know the number of terms, n. To find out how many terms there are in the sum, use the formula for arithmetic sequences:

$$\begin{aligned}a_n &= a_1 + (n-1)d \\ 500 &= 2 + (n-1)2 \\ 500 &= 2 + 2n - 2 \\ 500 &= 2n \\ n &= 250.\end{aligned}$$

Then, use the formula for arithmetic sums:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right) = 250\left(\frac{2+500}{2}\right) = 250\left(\frac{502}{2}\right) = 250 \cdot 251 = 62750.$$

Problem 14. An employer in Connecticut has offered you a starting salary of \$40,000 per year with 3% annual increases. An employer in San Diego has offered you a starting salary of \$35,000 per year with 4% annual increases.
(a) (**6 points**) For the Connecticut job, what would be your total earnings over a 10-year period?

Answer. First we need to recognize what kind of problem this is.

- The salary in year 1 is $40,000$.
- The salary in year 2 is $40,000 \cdot 1.03$.
- The salary in year 3 is $(40,000 \cdot 1.03) \cdot 1.03 = 40,000 \cdot 1.03^2$.
- Likewise, the salary in year n is $40000 \cdot 1.03^{n-1}$.

Since each year's salary is 1.03 times the previous year's salary, and since the question is to find the total earnings over a 10-year period, which is the sum of all the annual salaries, we have a geometric sum. Use the geometric-sum formula from page 1:

$$\begin{aligned} S_n &= a_1\left(\frac{1-r^n}{1-r}\right) \\ S_{10} &= a_1\left(\frac{1-r^{10}}{1-r}\right) \\ &= 40000\left(\frac{1-1.03^{10}}{1-1.03}\right) \\ &\approx 458,555. \end{aligned}$$

(b) (**6 points**) For the San Diego job, what would be your total earnings over a 10-year period?

Answer. Similar to the first part,

$$\begin{aligned} S_n &= a_1\left(\frac{1-r^n}{1-r}\right) \\ S_{10} &= a_1\left(\frac{1-r^{10}}{1-r}\right) \\ &= 35000\left(\frac{1-1.04^{10}}{1-1.04}\right) \\ &\approx 420,213. \end{aligned}$$

Problem 15. (**6 points**) As you rise through management, you will need to deal with other people's annoying problems more and more often. If you deal with 8 annoyances per month in your entry-level position, and if the number of annoyances per month increases by 20 percent for each level you rise in the company, how many annoyances per month can you expect once you reach regional vice president, which is the seventh layer of management?

Answer. Again, first there is the recognition problem: what kind of problem is this?

- At the first level of the company, you have 8 annoynances per month.

- At the second level of the company, you have $8 \cdot 1.20$ annoynances per month.
- At the third level of the company, you have $(8 \cdot 1.20) \cdot 1.2 = 8 \cdot 1.20^2$ annoynances per month.
- $\vdots$
- At the seventh level of the company, you have $8 \cdot 1.20^6$ annoynances per month.

This is a sequence of numbers, with each one 1.20 times the previous. That means it's a geometric sequence, and we just need to find the seventh term. Use the formula for geometric sequences:

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_7 &= a_1 r^{7-1} \\ &= 8 \cdot 1.20^6 \\ &\approx 24. \end{aligned}$$